Predicting the permeability function for unsaturated soils using the soil-water characteristic curve

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Received September 21, 1993
Accepted March 18, 1994

The coefficient of permeability for an unsaturated soil is primarily determined by the pore-size distribution of the soil and can be predicted from the soil-water characteristic curve. A general equation, which describes the soil-water characteristic curve over the entire suction range (i.e., from 0 to 10^6 kPa), was proposed by the first two authors in another paper. This equation is used to predict the coefficient of permeability for unsaturated soils. By using this equation, an evaluation of the residual water content is no longer required in the prediction of the coefficient of permeability. The proposed permeability function is an integration form of the suction versus water content relationship. The proposed equation has been best fit with example data from the literature where both the soil-water characteristic curve and the coefficient of permeability were measured. The fit between the data and the theory was excellent. It was found that the integration can be done from zero water content to the saturated water content. Therefore, it is possible to use the normalized water content (volumetric or gravimetric) or the degree of saturation data versus suction in the prediction of the permeability function.

Key words: coefficient of permeability, soil-water characteristic curve, unsaturated soil, water content, soil suction.

La coefficient de perméabilité d'un sol non saturé est principalement déterminé par la répartition de la taille des pores et il peut être prédit à partir de la courbe caractéristique sol-eau. Une équation générale décrivant la courbe caractéristique sol-eau sur la plage complète des valeurs de succion (soit de 0 à 10^6 kPa) a été proposée par les deux premiers auters dans un autre article. Cette équation est utilisée pour prédire le coefficient de perméabilité des sols non saturés. Grace à cette équation on n'a plus besoin d'évaluer le teneur en eau résiduelle pour prédire le coefficient de perméabilité. La fonction de perméablilité proposée est une forme intégral de la relation succion-teneur en eau. L'équation a été ajustée à partir d'examples de données recuillies dans la littérature ou l'on a mesuré à la fois la courbe caractéristique sol-eau et le coefficient de perméabilité. L'accord entre les données et le théorie est excellent. On a trouvé que l'intégration peut être faite pour des teneurs en eau allant de zéro à la teneur en eau de saturation. Il est donc possible d'utiliser la teneur en eau normalisée (volumique ou massique) on les données connues sur le degré de saturation en fonction de la succion pour prédire la fonction de perméabilité.

Mots clés : coefficient de perméabilité, courbe caractéristique sol-eau, sol non saturé, teneur en eau, succion du sol.

[Traduit par la rédaction]

Can. Geotech. J. 31. 533-546 (1994)

Introduction

There is no engineering soil property that can vary more widely than that of the coefficient of permeability. For saturated soils, the coefficient of permeability can vary more than 10 orders of magnitude when considering soils that range from a gravel to a clay. This wide range in coefficient of permeability has proven to be a major obstacle in analyzing seepage problems.

Soils that become desaturated are even more difficult to analyze. In this case it is possible for a single soil to have a coefficient of permeability that ranges over 10 orders of magnitude. Initial consideration of problems involving unsaturated soils might lead an engineer to conclude that no useful analyses are possible when the soil becomes unsaturated. However, experience has now shown that many important questions can be addressed using seepage analyses on unsaturated soils.

Generally an upper and lower bound can be established on the coefficient of permeability function for the soil. As a result, an upper and lower bound can also be established on other pertinent variables such as mass flows and pore-water pressures. Analyses have shown that the mass of water flowing through a soil is directly proportional to the coefficient of permeability. On the other hand, the pore-water pressures and hydraulic heads are to a large extent independent of the absolute coefficient of permeability values. This observation is particularly of value as analyses are extended into the unsaturated soil zone.

For many geotechnical problems involving unsaturated soils, a knowledge of the pore-water pressures or hydraulic heads is of primary interest. These values are relatively insensitive to the saturated coefficient of permeability and the permeability function. This becomes particularly valuable when one considers the difficulties of the work involved in measuring the unsaturated coefficients of permeability for a soil.

The coefficient of permeability has been shown to be a relatively unique function of the water content of a soil during the desorption process and a subsequent sorption process. The function appears to be unique as long as the volume change of the soil structure is negligible or reversible. However, to use this procedure in a seepage analysis, it is necessary to know the relationship between water content and matric suction. This relationship is highly hysteretic with respect to the desorption and sorption processes (Fig. 1). As a result, it appears to be more reasonable to go directly to a permeability function related to matric suction.
There is one permeability function for the desorption process and another function for the sorption process in an unsaturated soil. Both functions have a similar characteristic shape and as such can be fitted with a similar form of mathematical equation. Most engineering problems usually involve either a desorption process or a sorption process.

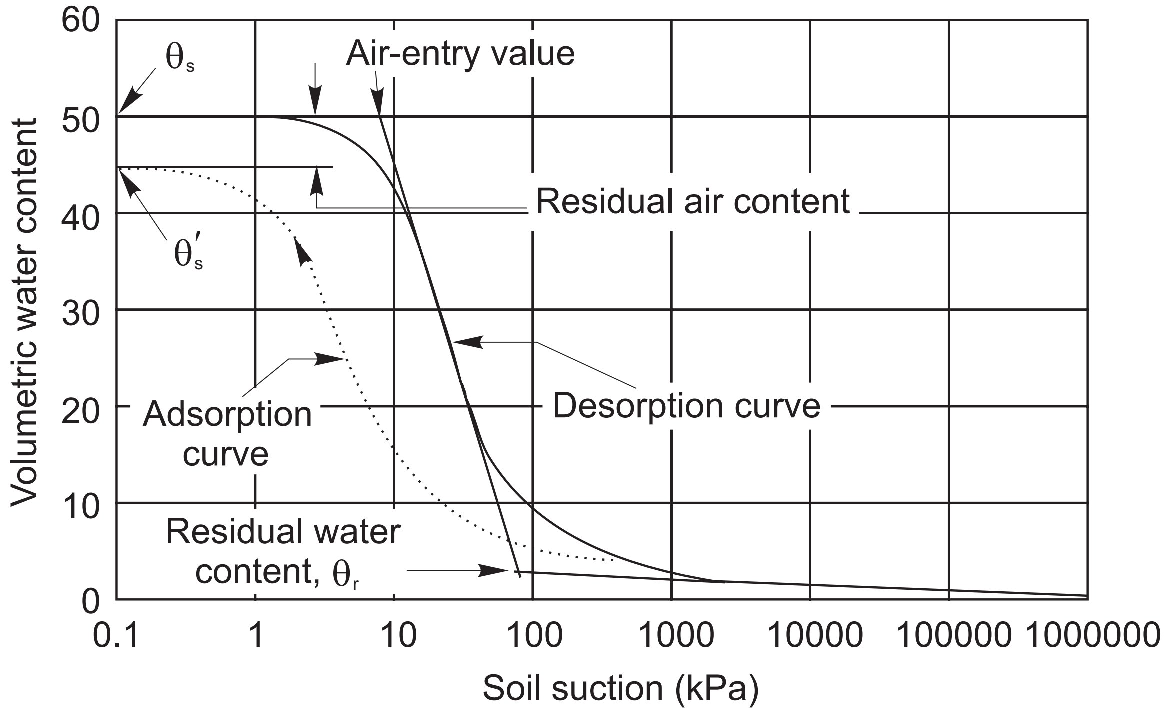

FIG 1. Typical Desorption and absorption curves for a silty soil. θ_s, saturated volumetric water content; θ_s‘.

Even in many cases where both desorption and sorption processes are involved, a single equation is appropriate for engineering purposes.

Numerous attempts have been made to predict empirically the permeability function for an unsaturated soil. These procedures make use of the saturated coefficient of permeability and the soil-water characteristic curve for the soil. As more precise equations have been developed for the soil-water characteristic curve, likewise, more reliable predictions have been made for the coefficient of permeability function.

This paper reviews the background of the empirical prediction of the coefficient of permeability function. Using the present knowledge of a mathematical relationship for the soil-water characteristic curve, a new permeability function is predicted. The theoretical basis for the permeability function is shown. The function is based on the entire soil-water characteristic curve (i.e., from saturation down to a water content of zero or a suction of 10^6 kPa) and assumes that the ease of water flow through the soil is a function of the amount of water in the soil matrix.

Definitions

The coefficient of permeability k of an unsaturated soil is not a constant. The coefficient of permeability depends on the volumetric water content θ, which, in turn, depends upon the soil suction, ψ. The soil suction may be either the matric suction of the soil, (i.e., u_a-u_w, where u_a is pore-air pressure, and u_w is pore-water pressure), or the total suction (i.e., matric plus osmotic suctions). Soil suction is one of the two stress state variables that control the behaviour of unsaturated soils. Therefore, it is suggested that the term “permeability function for unsaturated soils” be used to represent the relationship between the coefficient of permeability and soil suction. When the coefficient of permeability at any soil suction, $k(\psi)$, is referenced to the saturated coefficient of permeability k_s, the relative coefficient of permeability, $k_r(\psi)$, can be written as follows:

Table 1. Empirical equations for the unsaturated coefficient of permeability $k(\theta)$.

Function	Reference
$k_r = \Theta^n$, where $\Theta = (\theta - \theta_r)/(\theta_s - \theta_r)$, and n = 3.5	Averjanov (1950)
$k = k_s \left(\dfrac{\theta}{\theta_s} \right)^n$	Campbell (1973)
$k = k_s \exp[\alpha(\theta - \theta_s)]$	Davidson *et al.* (1969)

[1] $$k_r(\psi) = \frac{k(\psi)}{k_s}$$

The relative coefficient of permeability as a function of volumetric water content, $k_r(\theta)$, can be defined similarly. The relative coefficient of permeability, ($k_r(\psi)$ or $k_r(\theta)$), is a scalar function. The volumetric water content, θ, can be used in its normalized form, which is also referred to as the relative degree of saturation:

[2] $$\Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r}$$

where:

Θ= the nomalized volumetric water content or relative degree of saturation,

θ_s= the saturated volumetric water content, and

θ_r= the residual volumetric water content.

Degree of saturation, S, which indicates the percentage of the voids filled with water, is often used in place of the normalized water content, Θ.

Literature review

There are two approaches to obtain the permeability function of an unsaturated soil: (i) empirical equations, and

Table 2. Empirical equations for the unsaturated coefficient of permeability $k(\psi)$.

Function	Reference
$k = k_s, \quad \text{for} \quad \psi \le \psi_{aev}$	Brooks and Corey (1964)
$k_r = (\psi / \psi_{aev})^{-n} \quad \text{for} \quad \psi \ge \psi_{aev}$	
$k_r = \exp(-\alpha\psi)$	Gardner (1958)
$k = k_s / (a\psi^n + 1)$	
$k = a\psi + b$	Richards (1931)
$k = k_s \quad \text{for} \quad \psi \le \psi_{aev}$	Rijtema (1965)
$k_r = \exp[-\alpha(\psi - \psi_{aev})] \quad \text{for} \quad \psi_{aev} \le \psi \le \psi_1$	
$k = k_1 \left(\frac{\psi}{\psi_1}\right)^{-n} \quad \text{for} \quad \psi > \psi_1$	
$k = \alpha\psi^{-n}$	Wind (1955)

Notes: ψ_1 is the residual soil suction (i.e., ψ_r), and k_1 is the coefficient of permeability at $\psi = \psi_1$.

(ii) statistical models. Several measured permeability data are required to use an empirical equation. A statistical model can be used to predict the permeability function when the saturated coefficient of permeability, k_s, and the soil-water characteristic curve are available.

Empirical equations

Several empirical equations for the permeability function of unsaturated soils are listed in Tables 1 and 2. These equations can be used in engineering practice when measured data are available for the relationship between the coefficient of permeability and suction, $k(\psi)$, or for the relationship between the coefficient of permeability and the water content, $k(\theta)$. The smallest number of measured points required to use one of the permeability equations in Tables 1 and 2 is equal to the number of fitting parameters in the adopted equation. When the number of measurements exceeds the number of the fitting parameters, a curve-fitting procedure can be applied to determine the fitting parameters. This approach allows a closed-form analytical solution for unsaturated flow problems.

Statistical models

Statistical models have also been used to determine the permeability function for an unsaturated soil using the characteristics of the soil-water characteristic curve. This approach is based on the fact that both the permeability function and the soil-water characteristic curve are primarily determined by the pore-size distribution of the soil under consideration. Figure 2 shows a typical soil-water characteristic curve for a sandy loam and its permeability function. Based on the pore-size distribution, Burdine (1953) proposed the following equation for the relative coefficient of permeability:

$$[3] \quad k_r(\theta) = \frac{k(\theta)}{k_s} = \Theta^q \frac{\int_{\theta_r}^{\theta} \frac{d\theta}{\psi^2(\theta)}}{\int_{\theta_r}^{\theta_s} \frac{d\theta}{\psi^2(\theta)}}$$

where: $q = 2$. The square of the normalized water content was used to account for tortuosity. This model is significantly more accurate than the same equation without the correction factor, Θ^q.

Childs and Collis-George (1950) proposed a model for predicting the coefficient of permeability based on the random variation of pore size. This model was improved by Marshall (1958) and further modified by Kunze *et al.* (1968). The calculations are performed by dividing the relation between volumetric water content and suction into n equal water-content increments (Fig. 3). The following permeability function has been slightly modified to use SI units and matric suction instead of pore-water pressure head:

$$[4] \quad k(\theta_i) = \frac{k_s}{k_{sc}} \frac{T_s^2 \rho_w g}{2\mu_w} \frac{\theta_s^p}{n^2} \sum_{j=i}^{m} \left[(2j + 1 - 2i)\psi_j^{-2}\right]; \quad i = 1, 2, ..., m$$

where:

$k(\theta_i)$ is calculated coefficient of permeability for a specified volumetric water content θ_i, corresponding to the i-th interval,

i is the interval number that increases with decreasing water content (for example, i=1 identifies the first interval that closely corresponds to the saturated water content θ_s, and i=m identifies the last interval corresponding to the lowest water content θ_L, on the experimental soil-water characteristic curve),

*j is a*counter from *i* to *m*,

k_{sc} is the calculated saturated coefficient of permeability,

T_s is the surface tension of water,

ρ_w is the water density,

g is the gravitational acceleration,

μ_w is the absolute viscosity of water,

p is a constant that accounts for the interaction of pores of various sizes,

m is the total number of intervals between the saturated volumetric water content θ_s and the lowest water content θ_L on the experimental soil-water characteristic curve,

n is the total number of intervals computed between the saturated volumetric water content θ_s and zero volumetric water content (i.e., $\theta = 0$) (Note: $n = m\,[\theta_s/(\theta_s - \theta_L)]$, $m \le n$ and $m = n$, when $\theta_L = 0$), and,

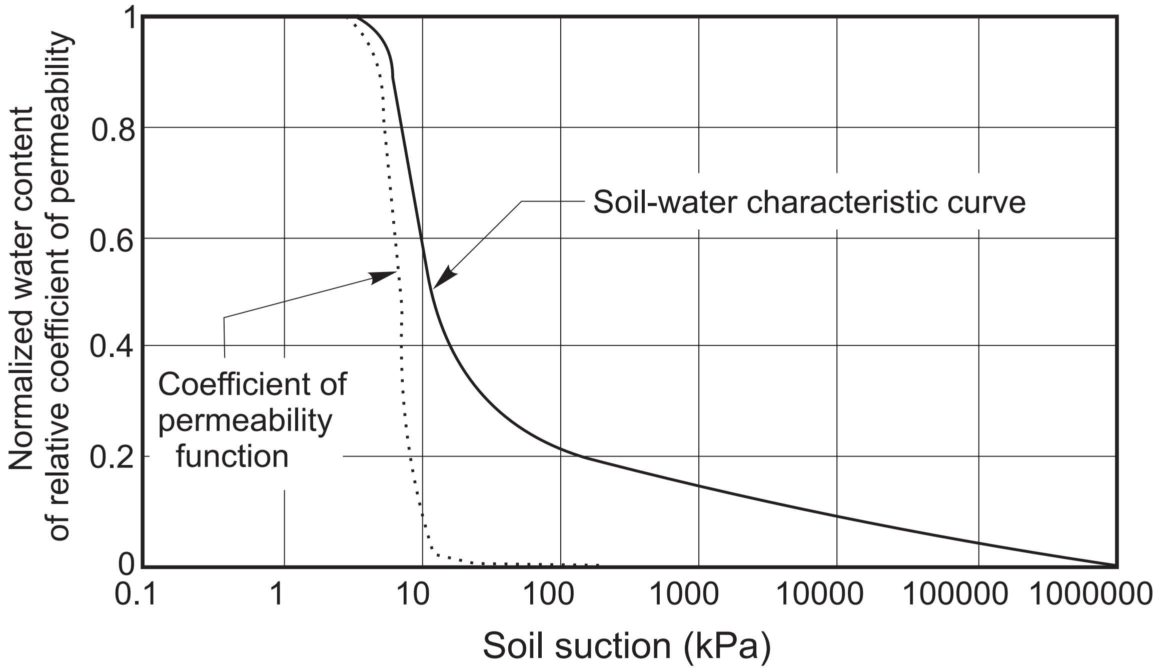

Fig. 2. Typical soil-water characteristic curve and permeability function for a silty soil.

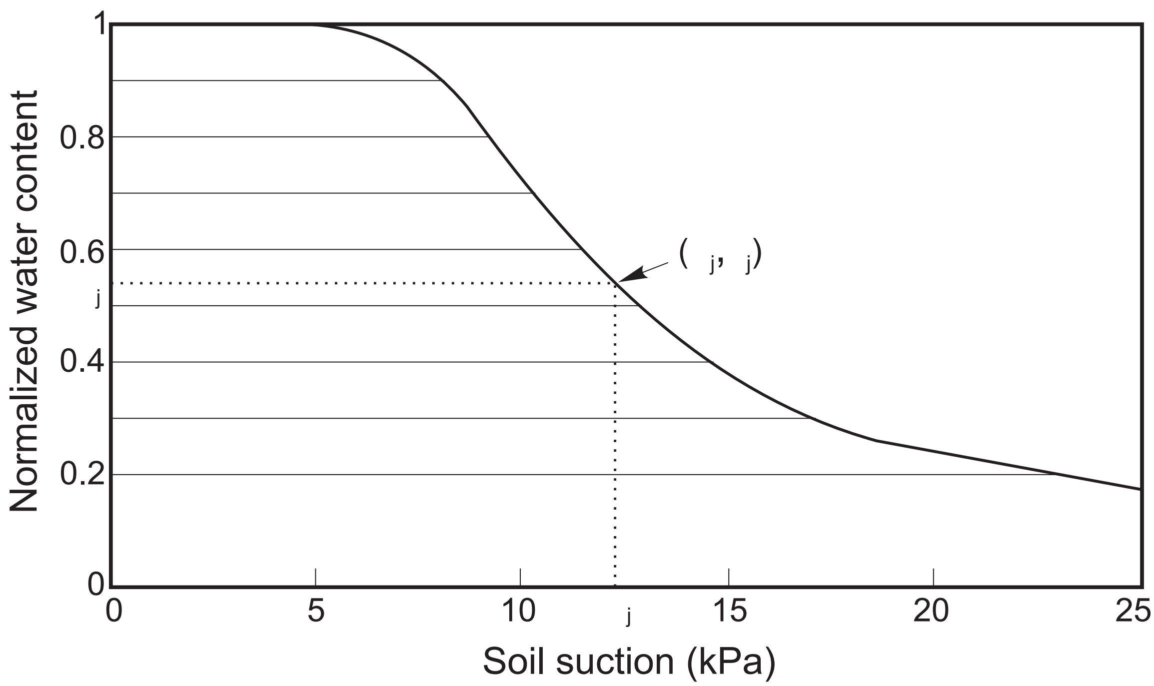

Fig. 3. Typical soil-water characteristic curve for predicting the permeability function. θ_j, midpoint of the j-th water content interval; ψ_j, suction corresponding to θ_j.

ψ_j is the suction (kPa) corresponding to the midpoint of the j-th interval (Fig. 3).

The calculation of the coefficient of permeability, $k(\theta_i)$, at a specific volumetric water content, θ_i, involves the summation of the suction values that correspond to the water contents at and below θ_i. The matching factor, (k_s / k_{sc}), based an the saturated coefficient of permeability, is necessary to provide a more accurate fit for the unsaturated coefficient of permeability. The shape of the permeability function is determined by the terms inside the summation-sign portion of the equation which, in turn, are obtained from the soil-water characteristic curve.

Mualem (1976*a*) analyzed a conceptual model of a porous medium similar to that of the Childs and Collis-George (1950) model and derived the following equation for predicting the coefficient of permeability:

$$[5] \qquad k_r(\theta)=\Theta^q\left(\frac{\int_{\theta_r}^{\theta}\frac{d\theta}{\psi(\theta)}}{\int_{\theta_r}^{\theta_s}\frac{d\theta}{\psi(\theta)}}\right)^2$$

where: $q = 0.5$. The value of q depends on the specific soil-fluid properties, and thus may vary considerably for different soils. Based on the permeability data of 45 soils, Mualem (1976*a*) found that the optimal value of q was 0.5.

When using a statistical model, soil suction has to be expressed as a function of volumetric water content. The integrations are performed along the volumetric water content axis and the permeability is expressed as a function of volumetric water content. Van Genuchten (1980) proposed an equation for the soil-water characteristic curve. By substituting his equation into the statistical models proposed by Burdine (1953) and Mualem (1976*a*) (i.e., Eqs. [3] and [5]), van Genuchten (1980) derived closed-form expressions for the permeability function.

Theory and computational formula

Both Eqs. [3] and [5] contain a correction factor, Θ^q, which depends upon the properties of the soil under consideration. This paper uses Eq. [4] for the prediction of the coefficient of permeability of unsaturated soils, since it was proven to give satisfactory predictions. It is assumed, throughout this paper, that the volume change of the soil structure is negligible. Equation [4] can be expressed in the form of an integration as follows:

$$[6] \qquad k(\theta)=\frac{k_s}{k_{sc}}\frac{T_s^2\rho_w g\theta_s^p}{\mu_w}\int_{\theta_L}^{\theta}\frac{\theta-x}{\psi^2(x)}dx$$

where:

ψ is soil suction which is given as a function of volumetric water content θ, and

x is a dummy variable of integration representing water content.

Equation [4] can also be expressed in the form of a relative coefficient of permeability, $k_r(\theta_i)$, as follows:

$$[7] \qquad k_r(\theta_i)=\sum_{j=i}^{m}\frac{2(j-i)+1}{\psi_j^2}\bigg/\sum_{j=i}^{m}\frac{2j-1}{\psi_j^2}$$

The integration form of Eq. [7] is as follows:

$$[8] \qquad k_r(\theta)=\int_{\theta_L}^{\theta}\frac{\theta-x}{\psi^2(x)}dx\bigg/\int_{\theta_L}^{\theta_s}\frac{\theta_s-x}{\psi^2(x)}dx$$

The residual volumetric water content, θ_r, is the water content below which a large increase in suction is required to remove additional water. The residual water content can generally be observed as a break in the soil-water characteristic curve. It is commonly assumed that the coefficient of permeability of a soil is essentially zero when its water content is below the residual water content. Kunze *et al.* (1968) investigated the effect of using a partial soil-water characteristic curve for the prediction of coefficient of permeability. It was concluded that the accuracy of the prediction is significantly improved when the complete soil-water characteristic curve is used namely when the condition,

$$[9] \qquad \theta=\theta_r, \qquad k_r=0$$

is realistically prescribed. The integration in Eq. [8], therefore, should be carried out in the interval θ_r to θ_s instead of the interval θ_L to θ_s:

$$[10] \qquad k_r(\theta)=\int_{\theta_r}^{\theta}\frac{\theta-x}{\psi^2(x)}dx\bigg/\int_{\theta_r}^{\theta_s}\frac{\theta_s-x}{\psi^2(x)}dx$$

Although all models for the relative coefficient of permeability k_r (i.e., Eqs. [3], [5], and [10]) require a knowledge of the residual water content, soil-water characteristic curves are rarely measured over their entire range of water content. Brooks and Corey (1964), Mualem (1976*a*, *b*), and van Genuchten (1980) suggest different methods for the extrapolation of the measured soil-water characteristic data and the determination of the residual water content. The conventional procedure for the prediction of the coefficient of permeability using Eq. [10] consists of two steps. First, the residual water content of the soil under consideration is estimated from the experimental data. Then, the measured soil-water characteristic data are fitted by a mathematical equation in the interval θ_r to θ_s, and the integrations are evaluated using the best-fit curve. A generally accepted procedure for determining the residual water content does not appear to exist. It would be an asset to predict the coefficient of permeability without having to estimate the residual water content.

The total suction corresponding to zero water content is essentially the same for all types of soils. This value is about 10^6 kPa and has been supported experimentally by results of tests on a variety of soils (Croney and Coleman 1961) and by thermodynamic considerations (Richards 1965). A general equation describing the soil-water characteristic curve over the entire suction range (i.e., 0 to 10^6 kPa) was proposed by Fredlund and Xing (1994):

$$[11] \qquad \theta=C(\psi)\frac{\theta_s}{\left\{\ln\left[e+(\psi/a)^n\right]\right\}^m}$$

where:

e is the natural number, 2.71828,
a is approximately the air-entry value of the soil,
n is a parameter that controls the slope at the inflection point in the soil-water characteristic curve,
m is a parameter that is related to the residual water content, and
$C(\psi)$ is a correcting function defined as

$$C(\psi)=1-\frac{\ln\left(1+\dfrac{\psi}{C_r}\right)}{\ln(1+1000000/C_r)}$$

where C_r is a constant related to the matric suction corresponding to the residual water content θ_r (Fredlund and Xing 1994). A typical value for residual suction is about 1500 kPa. Equation [11] fits the experimental data well in the entire suction range from 0 to 10^6 kPa (Fredlund and Xing 1994).

To calculate the coefficient of permeability using Eqs. [10] and [11], it is convenient to perform the integration along the soil suction axis. Equation [10] can be transformed into the following form:

$$[12] \qquad k_r(\psi)=\int_{\psi}^{\psi_r}\frac{\theta(y)-\theta(\psi)}{y^2}\theta'(y)\mathrm{d}y\bigg/\int_{\psi_{aev}}^{\psi_r}\frac{\theta(y)-\theta_s}{y^2}\theta'(y)\mathrm{d}y$$

Table 3. Soil properties and fitting parameters of Eq. [11] for the example soils.

Soil type	θ_s	k_s x 10^{-6} (m/s)	*a*	*n*	*m*	C_r
Touchet silt loam (GE3)	0.43	—	8.34	9.90	0.44	30.0
Columbia sandy loam	0.458	—	6.01	11.86	0.36	30.0
Yolo light clay	0.375	0.123	2.70	2.05	0.36	100.0
Guelph loam						
drying	0.52	3.917	5.61	2.24	0.40	300.0
wetting	0.43	—	3.12	4.86	0.23	100.0
Superstition sand	—	18.3	2.77	11.20	0.45	300.0

where:

ψ_{aev} is the air-entry value of the soil under consideration (i.e., the suction where air starts to enter the largest pores in the soil),
ψ_r is the suction corresponding to the residual water content θ_r,
y is a dummy variable of integration representing suction, and
θ' is the derivative of Eq.[11].

Since Eq. [11] fits the experimental data over the entire suction range, the integrations in Eq. [12] can be performed from ψ_{aev} to 10^6 for all types of soils. This greatly simplifies the prediction procedure for the coefficient of permeability, since the residual value (θ_r or ψ_r) does not have to be determined experimentally for each soil.

To avoid the numerical difficulties of performing the integration over the soil suction range from ψ_{aev} to 10^6 kPa, it is more convenient to perform the integration on a logarithmic scale. Therefore, the following variation of Eq. [12] is preferred:

[13]
$$k_r(\psi) = \frac{\int_{\ln(\Psi)}^{b} \frac{\theta(e^y) - \theta(\psi)}{e^y} \theta'(e^y) dy}{\int_{\ln(\Psi_{aev})}^{b} \frac{\theta(e^y) - \theta_s}{e^y} \theta'(e^y) dy}$$

where:
b = ln(l000000), and
y is a dummy variable of integration representing the logarithm of integration.

When the soil-water characteristic curve is considered over the entire suction range, volumetric water content is referenced to zero water content (otherwise, the normalized water content becomes negative if θ is less than θ_r**).** In this case, the normalized volumetric water content is equivalent to the degree of saturation, provided that the volume change of the soil structure is negligible:

[14]
$$\Theta = \frac{\theta}{\theta_s} = \frac{V_w / V_t}{V_v / V_t} = \frac{V_w}{V_v} = S$$

where:
V_w is the volume of water in the soil specimen,
V_v is the volume of void in the soil specimen, and
V_t is the total volume of the soil specimen.

Similarly, if the gravimetric water content *w* is referenced to zero water content, the normalized gravimetric water content *W* can be written as follows:

[15]
$$W = \frac{w}{w_s} = \frac{W_w / W_p}{W_{ws} / W_p} = \frac{W_w}{W_{ws}} = \frac{V_w \gamma_w}{V_v \gamma_w} = \frac{V_w}{V_v} = S$$

where:
w_s is the saturated gravimetric water content,
W_w is the weight of water,
W_p is the weight of soil particles,
W_{ws} is the weight of water when the soil is at saturation. and
γ_w is the specific weight of water.

Equations [14] a [15] indicate that the normalized (gravimetric or volumetric) water content is identical to the degree of saturation when referenced to zero water content and the volume change of the soil structure is negligible. Therefore Eq. [13] is valid for any of the three water-content variables, (i.e., volumetric water content, gravimetric water content, or degree of saturation) if the soil-water characteristic curve is described by Eq.[11].

Numerical results and comparisons with experimental data

Numerical integration using [13] has been incorporated into a curve-fitting program (written in C language) called CFVIEW. The program first determines the four parameters *a*, *n*, *m*, and C_r in Eq. [11] through use of a non-linear least squares routine. With the soil-water characteristic curve known, the program then calculates the permeability function using Eq. [13]. The procedure of numerical integration for Eq. [13] is listed in the Appendix. The best-fit analysis is done on the suction versus water-content data. None of the residual values θ_r, and C_r is explicitly required during the curve fitting procedure. The detailed non-linear curve-fitting algorithm was outlined by Fredlund and Xing (1994). Comparisons are given in this section of the paper between measured and predicted coefficient of permeability curves for five soils. The soil properties and values of the fitting parameters for Eq. [11] for each soil are listed in Table 3.

Figure 4 shows a best-fit curve to the experimental data for Touchet silt loam (GE3) from Brooks and Corey (1964). The predicted coefficient of permeability based on the best fit curve in Fig. 4 is compared with the measured permeability data in Fig. 5. The predicted curve is close to the measured permeability data. Results obtained for Columbia sandy loam (Brooks and Corey 1964) are presented in Figs. 6 and 7. Good predictions of the relative coefficient of permeability are obtained.

A best-fit curve to the experimental data for Superstition sand (Richards 1952) is shown in Fig. 8. The relative coefficient of permeability predicted based on its soil-water characteristic curve is shown in Fig. 9. It can be seen that the

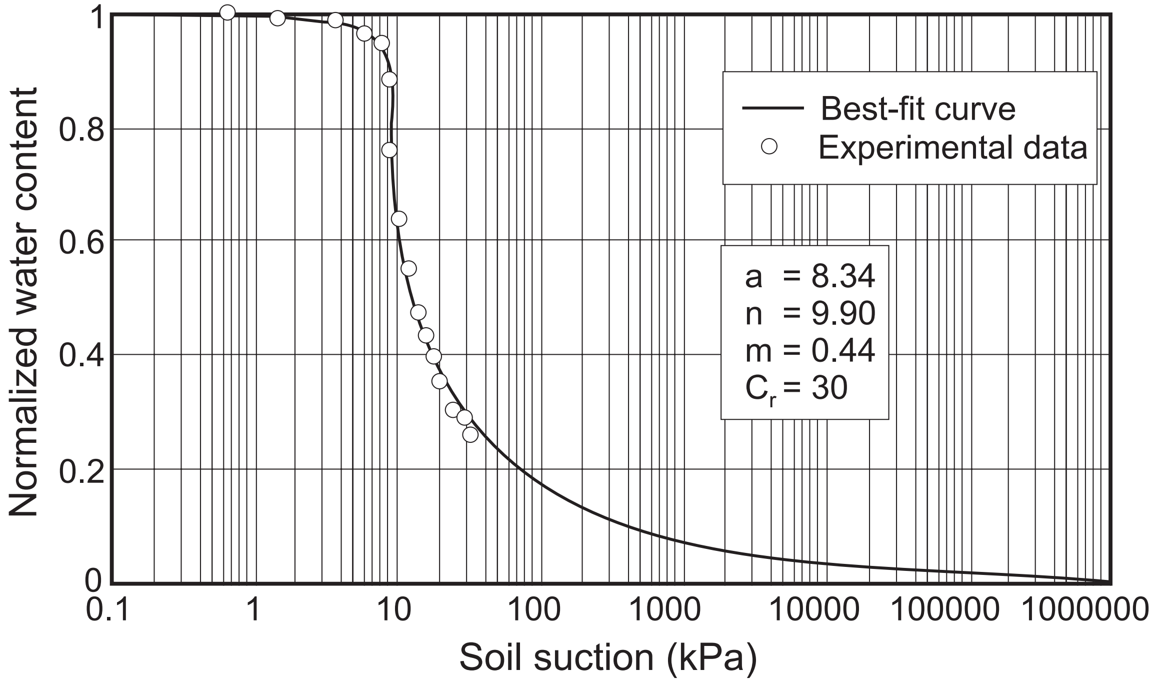

Fig. 4. Best-fit curve to the experimental data for Touchet silt loam (GE3) (data from Brooks and Corey 1964).

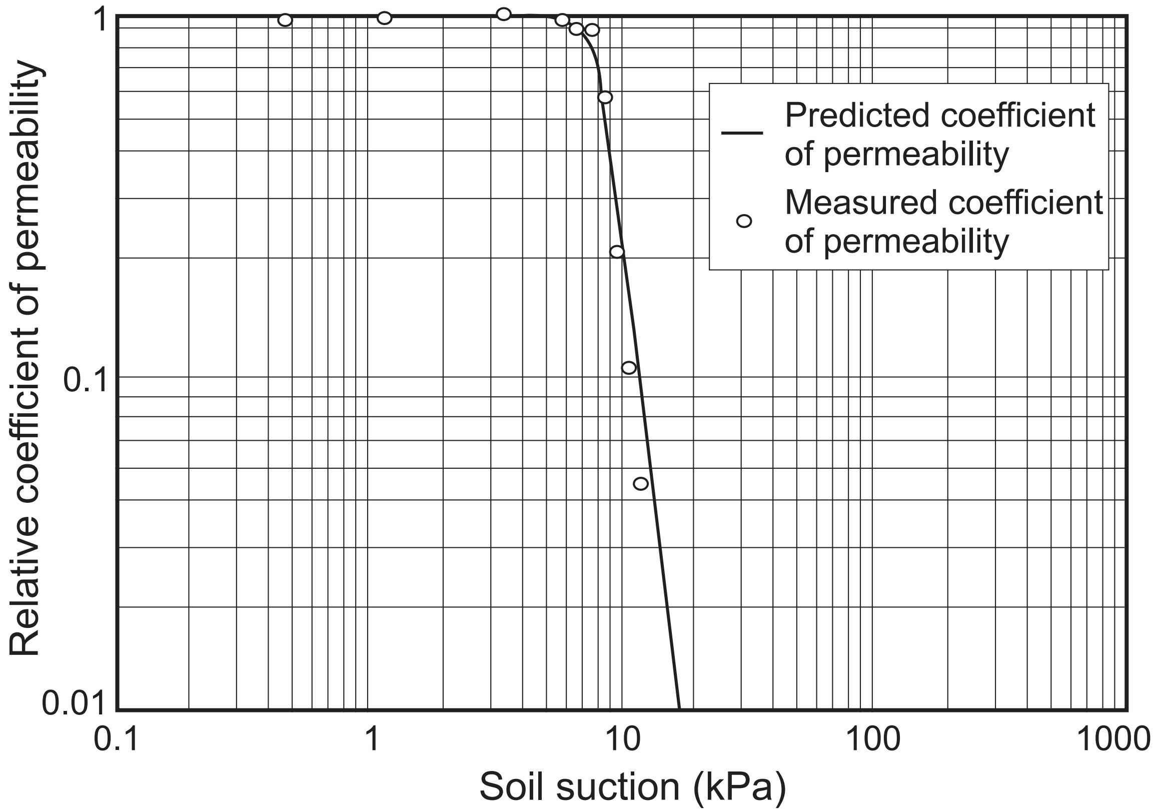

Fig. 5. Comparison of the predicted relative coefficient of permeability with the measured data for Touchet silt loam (GE3) (data from Brooks and Corey 1964).

prediction of the relative coefficient of permeability becomes less accurate as the soil suction increases. This poor prediction can be traced back to the poor fit to the experimental data as shown in Fig. 8. The poor fit to the experimental data is partially caused by the limitation of the curve-fitting program. A minor modification to the curve-fitting program gives a better fit of the experimental data. The improved best-fit curve to the experimental soil-water characteristic data for the Superstition sand is shown in Fig. 10. Compared with the results shown in Fig. 9, the prediction based on the improved best-fit curve is considerably more accurate, as shown in Fig. 11. Therefore, a good description of the experimental soil-water characteristic data is essential to obtain an accurate prediction of the coefficient of permeability.

The results for Yolo light clay (Moore 1939) are presented in Figs. 12 and 13. The predicted permeability is close to the measured data when the soil suction is less than 4 kPa.

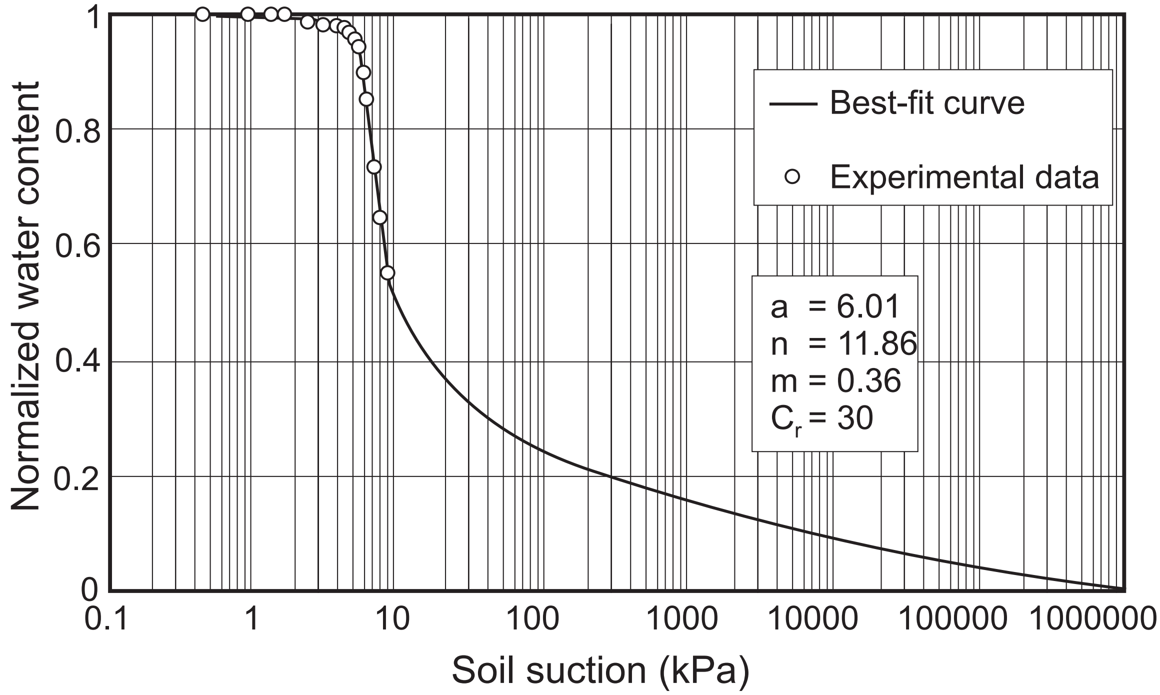

Fig. 6. Best-fit curve to the experimental data for Columbia sandy loam (data from Brooks and Corey 1964).

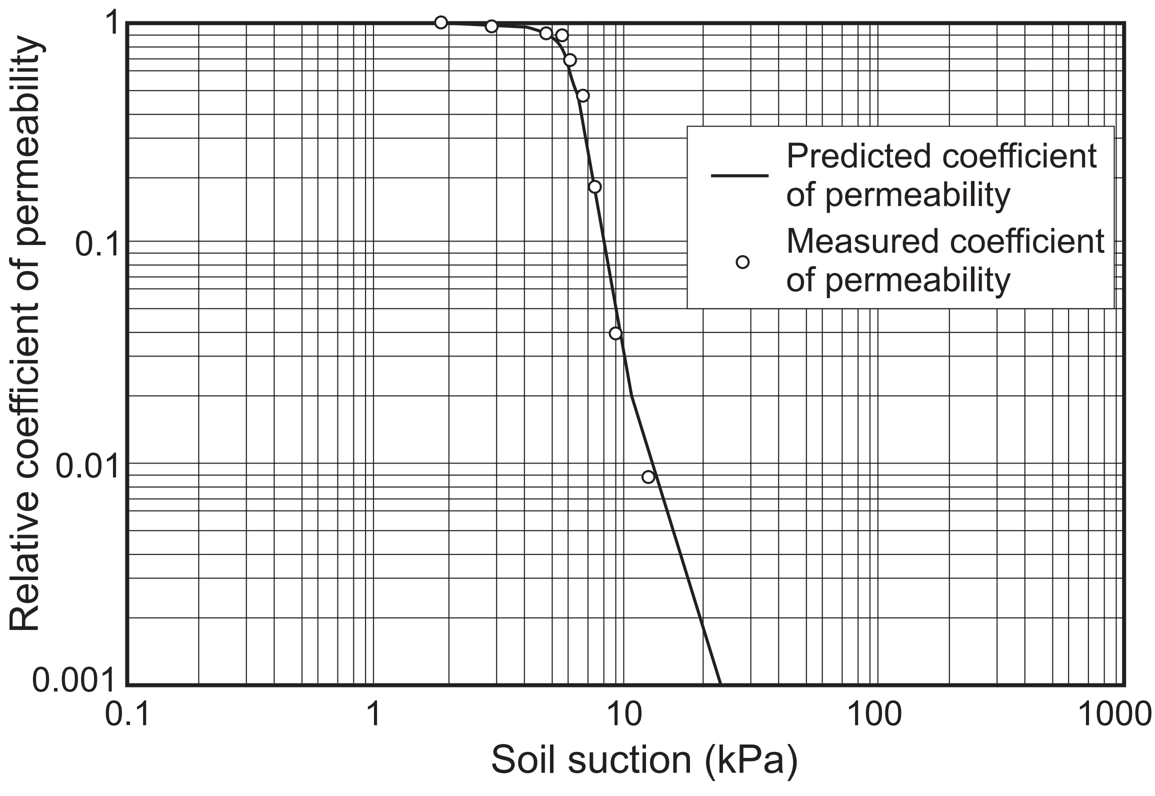

Fig. 7. Comparison of the predicted relative coefficient of permeability with the measured data for Columbia sandy loam (data from Brooks and Corey 1964).

It would appear that there may have been errors involved in the measured permeability data when the suction was greater than 4 kPa. The prediction of coefficient of permeability for clayey soils is generally less accurate than that for sandy soils.

Results for Guelph loam (Elrick and Bowman 1964) are shown in Figs. 14 and 15. This is an example in which hysteresis is present in the soil-water characteristic curve. As shown by the results, although a considerable hysteresis loop is observed in the $\theta(\psi)$ plane, the $k(\theta)$ hysteresis is much less significant. Therefore, in most cases, only one branch (normally the drying branch) is used in the calculation of permeability.

Summary of fit between theory and data

All the experimental soil-water characteristic data used in this paper are in the low suction range. Equation [11] is not only used to fit the experimental data in the low suction

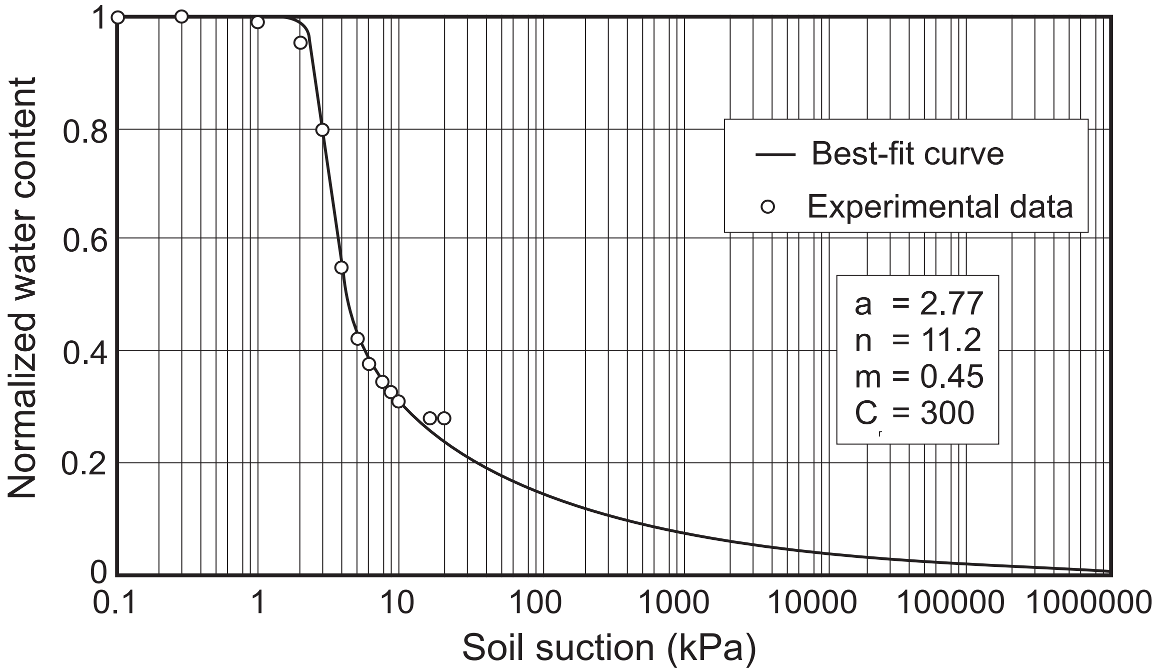

Fig. 8. . Best-fit curve to the experimental data for Superstition sand (data from Richards 1952).

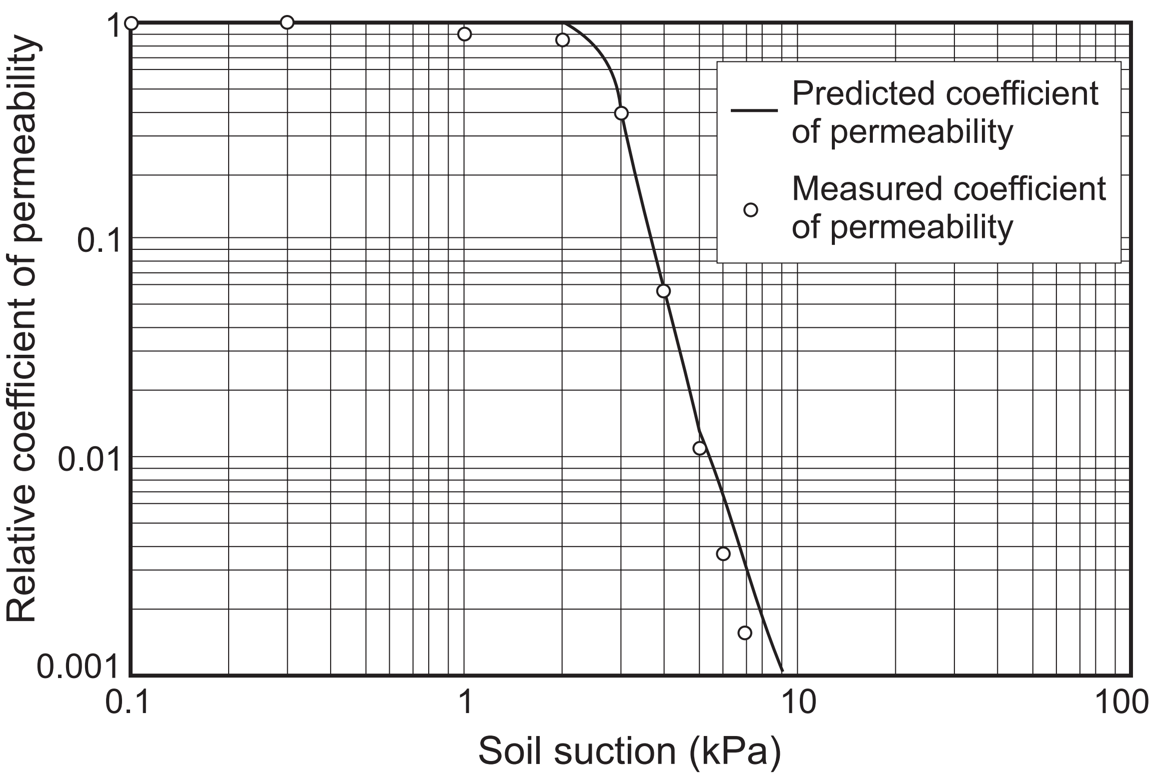

Fig. 9. Comparison of the measured data with the predicted relative coefficient of permeability using the best-fit curve in Fig. 8 (data from Richards 1952).

range, but also to estimate the soil-water characteristic behavior in the high suction range. By so doing, it is no longer necessary to estimate the residual volumetric water content θ_r for each soil when predicting the coefficient of permeability. Since reasonable predictions of the coefficient of permeability are obtained in all the cases studied in this paper, Eq. [11] gives a satisfactory description of the soil-water characteristic curve, which can then be used in the prediction of the coefficient of permeability. It is recommended, however, that the soil-water characteristic curve also be measured in the high suction range whenever possible, in order to increase the accuracy in the prediction of the coefficient of permeability.

The accuracy of the prediction of the coefficient of permeability depends not only on the closeness of the best-fit curve to the experimental soil-water characteristic data, but also on the precision of the model adopted. As concluded by Mualem (1986), there is no single model that fits every soil. The proposed models have been found to be most satisfactory

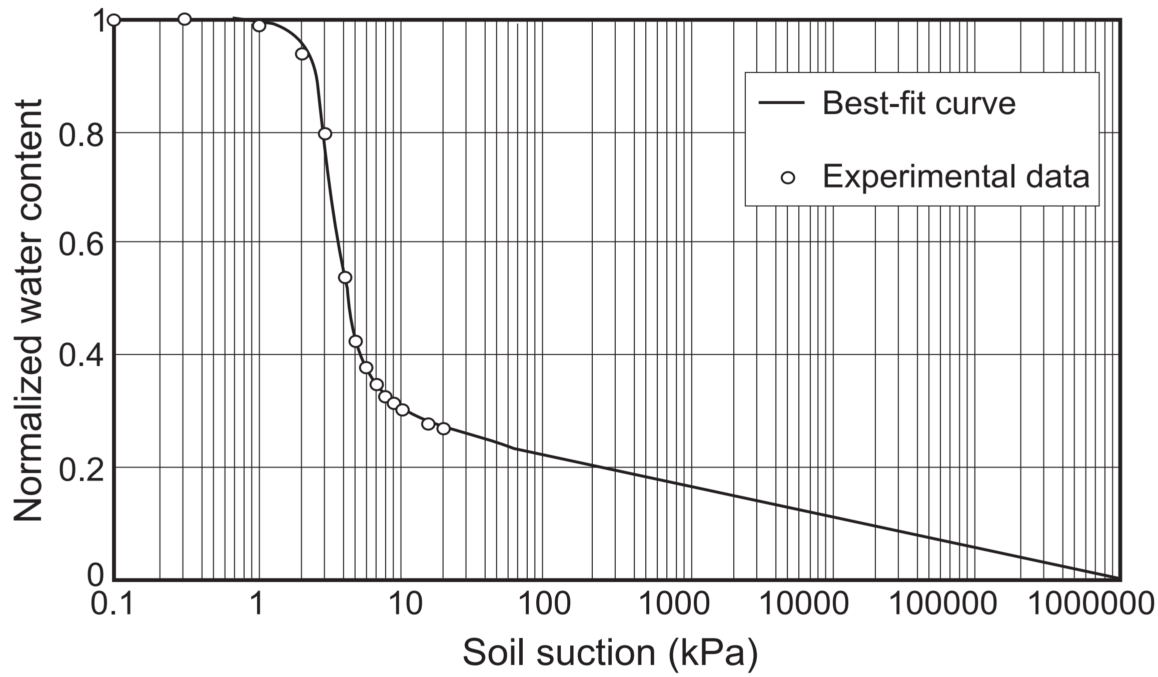

Fig. 10. Improved best-fit curve to the experimental data shown in Fig. 8.

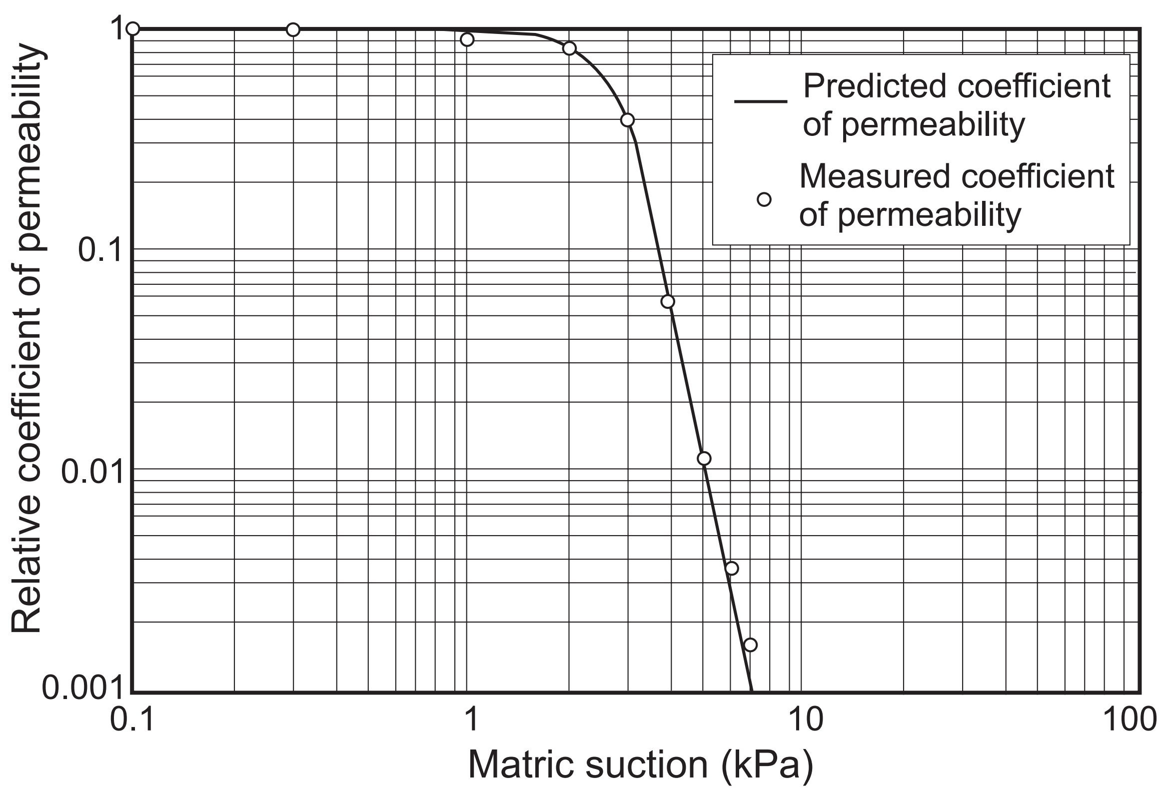

Fig. 11. Comparison of the measured data with the predicted relative coefficient of permeability using the improved best-fit curve in Fig. 10.

for sandy soils, whereas agreement with experimental data is often unsatisfactory for fine-grained soils. The flexibility of Eq. [13] can be increased by multiplying the equation by a correction factor, Θ^q. The correction factor takes into account the tortuosity (Mualem 1986). More accurate predictions can be obtained by using the following modification of Eq. [13]:

$$[16]\quad k_r(\psi) = \Theta^q(\psi)\,\frac{\displaystyle\int_{\ln(\psi)}^{b} \frac{\theta(e^y) - \theta(\psi)}{e^y}\,\theta'(e^y)\,dy}{\displaystyle\int_{\ln(\psi_{aev})}^{b} \frac{\theta(e^y) - \theta_s}{e^y}\,\theta'(e^y)\,dy}$$

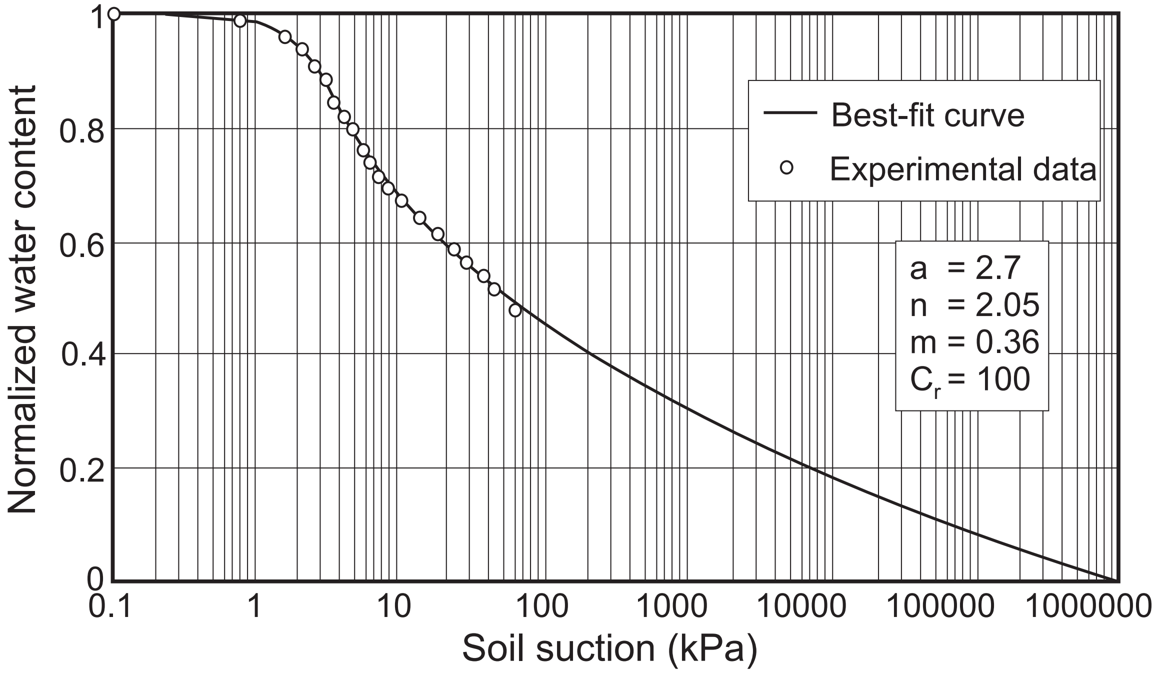

Fig. 12. Best-fit curve to the experimental data for Yolo light clay (data from Moore 1939).

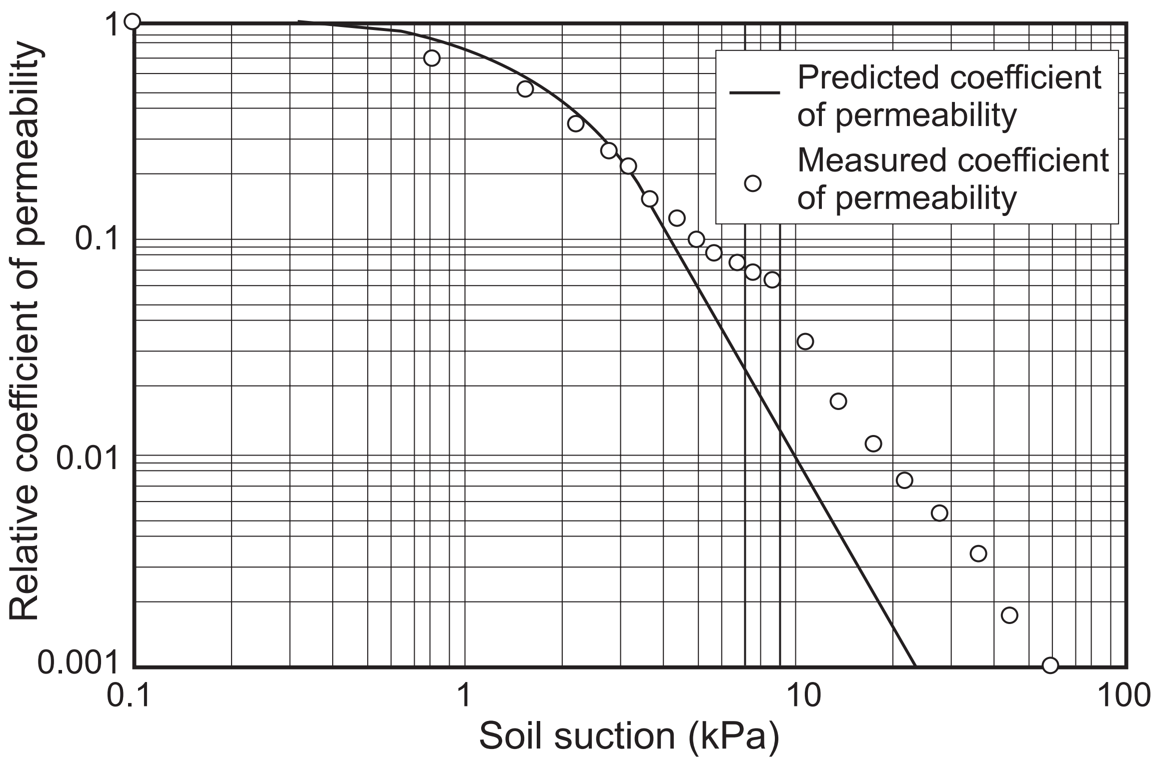

Fig. 13. Comparison of the predicted relative coefficient of permeability with the measured data for Yolo light clay (data from Moore 1939).

Based on the work by Kunze *et al*. (1968), the value of the power q is equal to 1.

Application in engineering practice

The flow of water through saturated-unsaturated soils is of primary significance is an increasing number of engineering problems. The design of earth dams or tailings impoundments require flow models to describe the flow in the saturated-unsaturated zones under various boundary and soil conditions. Resource development and waste management are two examples where a rational approach to seepage analysis is needed. Recent developments in the area of unsaturated soils offer the appropriate concepts and technology for a detailed study of seepage through unsaturated soils. With the aid of digital computers, numerical methods, such as the finite-difference and finite-element methods, can be applied to solve the complex differential equations for saturated-unsaturated flow.

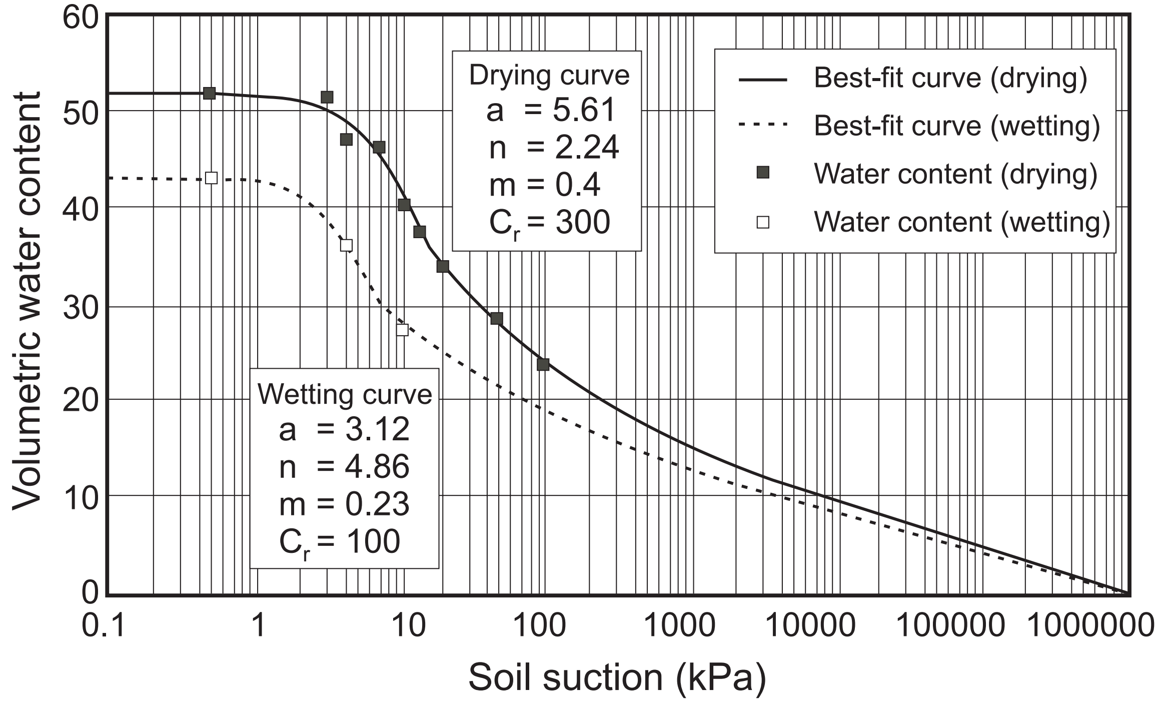

Fig. 14. Best-fit curves to the experimental data for Guelph loam (data from Elrick and Bowmann 1964).

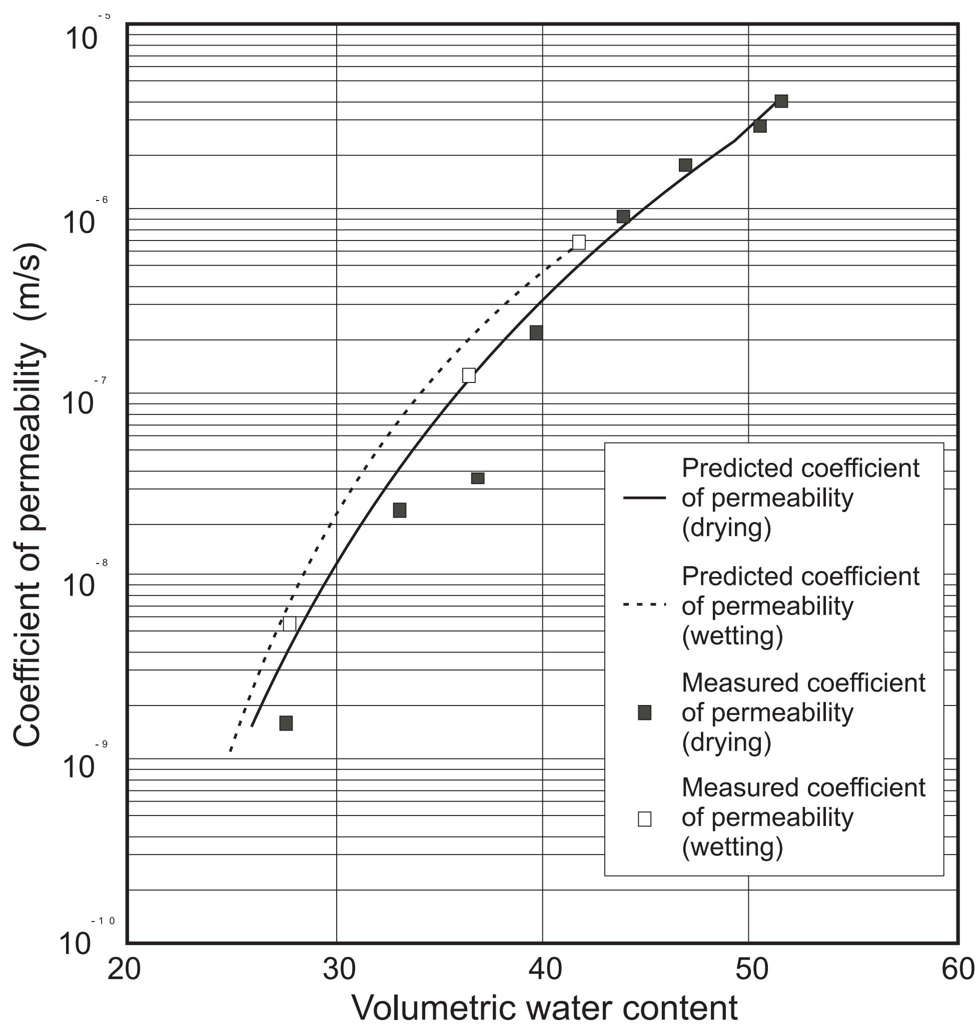

Fig. 15. Comparison of the predicted coefficient of permeability with the measured data for Guelph loam (data from Elrick and Bowmann 1964).

The fundamental equation that governs the flow of water in a saturated-unsaturated soil is as follows:

$$[17]\quad \frac{\partial}{\partial x}\left(k_x(\psi)\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y(\psi)\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z(\psi)\frac{\partial h}{\partial z}\right)+q=\frac{\partial\theta}{\partial t}$$

where:
h is the total fluid head,
k_x, k_y, and k_z are the material properties (permeability) in the x, y, and z directions, respectively,
q is the applied external boundary flux,
θ is the volumetric water content, and
t is the time.

To model the flow of water in an unsaturated soil, it is necessary to define the relationship between the coefficient of permeability and soil suction. One method of defining this relationship is to use one of the empirical equations listed in Tables 1 and 2 (for example, the equation proposed by Gardner 1958). As pointed out earlier, this method requires measured data for the permeability function to determine the parameters in the permeability equation. Since establishing the soil-water characteristic curve is generally not as difficult nor demanding as measuring the coefficient of permeability at various suction levels, the task of defining the permeability function can be simplified by calculating the coefficient of permeability from the soil-water characteristic curve using the procedure proposed in this paper. In other words, the prediction procedure proposed in this paper has become part of the seepage modeling program.

Conclusions

The numerical results predicted using the procedure proposed in this paper show good agreement with the published data on measured coefficients of permeability for unsaturated soils. The equation for the soil-water characteristic curve proposed by Fredlund and Xing (1994) was found to be effective in the prediction of the coefficient of permeability for unsaturated soils. In the case where experimental soil-water characteristic data are not available in the high-suction range, this equation can be used to estimate the soil-water characteristic behaviour in this range.

The procedure proposed in this paper for predicting the unsaturated coefficient of permeability does not require a knowledge of the residual water content of the soil under consideration. Since the residual water content is assumed to be zero for all soils, the normalized water content is identical to the degree of saturation and, therefore, the proposed procedure for the prediction of the unsaturated coefficient of permeability is valid for all three variables describing water content in a soil (i.e., volumetric water content, gravimetric water content, and degree of saturation). The predicted curve for the coefficient of permeability is sensitive to the soil-water characteristic curve. Therefore, an accurate description of the soil-water characteristic behavior is essential to the prediction of the coefficient of permeability. A modification of the procedure (i.e., Eq. [16]) is suggested to increase the accuracy of the prediction of the coefficient of permeability. This modification combines Eq. [13] and the empirical equation proposed by Averjanov (1950) in Table 1.

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Appendix: Numerical integration of Eq. [13] for the relative coefficient of permeability

The permeability function is expressed in the form of integrations that are performed on a logarithm scale (see Eq. [13]). The two integrals in Eq. [13] can be evaluated using the following numerical integration method.

Let a and b denote the lower and upper limits of the integration. Then,

[A1] $$a = \ln(\psi_{aev}), \quad b = \ln(1000000)$$

Let us divide $[a, b]$ into N subintervals of the same size and let Δy denote the length of each subinterval. Then,

[A2] $$a = y_1 < y_2 < \ldots < y_N < y_{N+1} = b, \quad \Delta y = \frac{b-a}{N}$$

The denominator of Eq. [13] can be evaluated as follows:

[A3] $$\int_{\ln(\psi_{aev})}^{b} \frac{\theta(e^y) - \theta_s}{e^y}\theta'(e^y)\,dy \approx \Delta y \sum_{i=1}^{N} \frac{\theta(e^{\bar{y}_i}) - \theta_s}{e^{\bar{y}_i}}\theta'(e^{\bar{y}_i})$$

where $\bar{y}_i$ is the midpoint of the i-th interval, $[y_i, y_{i+1}]$, and θ' is the derivative of Eq. [11], which is given as follows:

[A4] $$\theta'(\psi) = C'(\psi)\frac{\theta_s}{\left\{\ln\left[e + (\psi/a)^n\right]\right\}^m} - C(\psi)\frac{\theta_s}{\left\{\ln\left[e + (\psi/a)^n\right]\right\}^{m+1}}\frac{mn\left(\frac{\psi}{a}\right)^{n-1}}{a\left[e + (\psi/a)^n\right]}$$

where:
$$C'(\psi) = \frac{-1}{(C_r + \psi)\ln\left(1 + \frac{100000}{C_r}\right)}$$

For any suction value ψ in the range between the air-entry value ψ_{aev} and 10^6, the value $\ln(\psi)$ is between a and b. Assume that $\ln(\psi)$ is in the j-th interval, $[y_j, y_{j+1}]$. Then, the numerator of Eq. [13] can be evaluated as follows:

[A5] $$\int_{\ln(\psi)}^{b} \frac{\theta(e^y) - \theta(\psi)}{e^y}\theta'(e^y)\,dy \approx \Delta y \sum_{i=j}^{N} \frac{\theta(e^{\bar{y}_i}) - \theta(\psi)}{e^{\bar{y}_i}}\theta'(e^{\bar{y}_i})$$

Therefore, the relative permeability $k_r(\psi)$ at suction ψ is:

[A6] $$k_r(\psi) \approx \sum_{i=j}^{N} \frac{\theta(e^{\bar{y}_i}) - \theta(\psi)}{e^{\bar{y}_i}}\theta'(e^{\bar{y}_i}) \Bigg/ \sum_{i=1}^{N} \frac{\theta(e^{\bar{y}_i}) - \theta_s}{e^{\bar{y}_i}}\theta'(e^{\bar{y}_i})$$

The lower limit of the integration given by Eq. [13] corresponds to the saturated water content θ_s. Although the air-entry value is used as the lower limit of the integration, any other value between 0 and ψ_{aev} can be used. In other words, the air-entry value does not have to be known precisely. However, the value must be positive to perform the integration on a logarithm scale.