Analytical Methods for Slope Stability Analysis

D. G. Fredlund
University of Saskatchewan,
Saskatoon, Saskatchewan, Canada

Proceedings of the Fourth International Symposium on Landslides, State-of-the-Art,
pp. 229-250. September 16-21, Toronto, Canada. 1984

SYNOPSIS

During the past 7 years, research has been conducted on all aspects of limit type analyses and displacement type (finite element method) analyses for slope stability problems. The two-dimensional limit equilibrium method is most commonly used in practice. Difficulties associated with this type of analysis have been researched. An interslice force function has been proposed based on finite element analyses. The relationship between various limit equilibrium methods is now well understood.

Studies involving other methods have improved our understanding of slope stability problems. Limit (plasticity) upper and lower bound solutions have provided a useful context for comparing various methods of slope stability analysis. The variational calculus method allows the direct computation of the most critical slip surface for special cases.

INTRODUCTION

The primary objective of this paper is review and discussion of the main research studies that have been conducted during the past 7 years on slope stability analysis. A review of earlier research can be found in research papers such as Taylor (1937), Janbu (1973), and in textbooks such as Chowdhury (1978) and Chen (1975). The outline of the paper represents those topics that the author feels have been of most research interest. Most consideration is given to 2-dimensional analyses although some reference is made to 3-dimensional analyses.

The numerous approaches to slope stability analysis can be categorized as either i) limit (plasticity-type) formulation or ii) displacement formulations such as the finite element method (Wroth, 1976). Limit formulations provide a theoretical context for understanding the range of answers that can be expected from a slope stability analysis (Mendelson, 1968). These formulations are referred to as upper and lower bound solutions. The method of characteristics for stresses is an example of a lower bound solution. The method of characteristics for displacements is an example of an upper bound solution. The commonly used limit equilibrium methods are an upper bound type of solution. Recently the variational calculus technique has been combined with the limit equilibrium formulation in an attempt to directly compute the location of the critical slip surface. The probability theory has been applied to account for variations in measured quantities such as soil properties. The probability approach will be dealt with by Chowdhury, 1984.

Most emphasis is given in this paper to research associated with limit equilibrium methods. These methods have been best developed and tested by means of actual case histories. The limit equilibrium theory is briefly presented along with discussions on difficulties often encountered in solving for the factor of safety. A theoretical comparison is made of various limit equilibrium methods along with the assumptions made regarding some of the forces. Common techniques for solving the factor of safety equations are outlined.

One of the means to render the limit equilibrium analysis determinate is to assume an interslice, side force function. Recent research using the finite element method has resulted in the formulation of a generalized side force function defining the direction of the resultant interslice forces. The equation takes the form of a modified error function and is dependent on the geometry of the ground surface relative to the location of the slip surface.

Quantitative comparisons of computed factors of safety have been made by several researchers. The results are summarized and form a basis for deciding upon the method of analysis that should be used.

Other topics briefly discussed are i) the use of nonlinear shear strength envelopes, ii) analysis to account for earthquake loadings and iii) progressive failure of the soil. The selection of suitable soil parameters is outside the scope of this paper.

LOCATION OF THE CRITICAL SLIP SURFACE
A study of the stability of a slope generally involves the following steps: i) a survey of the elevation of the ground surface on a section perpendicular to the slope, ii) the advancement of several boreholes to identify the stratigraphy and obtain undisturbed soil samples, iii) the laboratory testing of the undisturbed soil samples to obtain suitable shear strength parameters for each stratigraphic unit, and iv) the installation of piezometers to measure the pore-water pressure. These steps provide the input data for performing a stability analysis. However, the location and shape of the most critical slip surface is an unknown (Figure 1). Some combination of acting and resisting forces along a slip surface of unknown shape and location will produce the lowest factor of safety. Of course, in the case of an already failed slope, the location of the slip surface is known.

FIG. 1 Slope Stability Example Showing Unknown Slip Surface Information

In design, the shape of the unknown slip surface is generally assumed while the location is determined by a trial and error procedure. If the shape of the slip surface is assumed to be circular, a grid of centers can be selected, and the radius varied at each center, providing a coverage of all possible conditions (Figure 2). When the slip surface takes on a composite shape (i.e., part circular and part linear) it is still possible to use a grid of centers and varying radius to search for a critical slip surface (Fredlund, 1981). This technique can also be used to generate slip surfaces of a wedge or sliding block configuration. Although the calculations can readily be performed using a computer, the costs can be excessive.

Various automatic search routines have been programmed to reduce the amount of computations. Some routines start at a point and seek the critical center by moving in a zigzag manner (Wright, 1974). Others use an initially coarse grid of centers which rapidly converges to the critical center (Fredlund, 1981).

Other search techniques have also been proposed. Celestino and Duncan (1981) have proposed a technique involving the minimization of multivariable functions (Figure 4). The technique can be used to locate a critical slip surface of any shape using a limit number of points (e.g., 4 to 6 points). One point is shifted until its optimum position is located and the same procedure is continued for other points. Two schemes, known as i) the alternating variable method and ii) the quadratic fit, have been used for performing the minimization.

FIG. 4 Shifting Points on the Slip Surface to Locate Critical Surface (From Celestino and Duncan, 1981)
Siegels, Kovacs and Lovell (1981) proposed a random surface generator to locate a critical slip surface of any shape (Figure 5). The procedure uses a random number generator and imposes restrictions on the shape and position of the slip surface to avoid nonsensical results. The slip surface consists of a series of straight lines with the following restrictions: i) the starting position on the boundary must be within a logical range, ii) the shape of the slip surface must be kinematically acceptable, iii) the base of the slip surface must lie above a lower boundary (i.e., strong layer), and, iv) the terminating position must lie within a logical range on the boundary. It is suggested that 50 to 200 slip surfaces should be generated in order to locate the most critical slip surface.

![Diagram of a sliding block generator using more than two boxes](From Siegel, Kovacs and Lovell, 1981)

The method of characteristics can either be an upper or lower bound solution in plasticity theory which can be used to locate the critical slip surface (Reddy, 1980; Sokolovskii, 1960). The soil at each and every point along the slip surface is assumed to be at the point of failure. A load is placed on the top of the slope to produce the failure condition. The location of the critical slip surface can be readily located in the case of simple, steep slopes. For flatter slopes with slip surfaces below the toe of the slope, the slip surface must be located by trial and error. This procedure is of academic interest but certainly does not provide a direct solution for the critical slip surface for the variety of geometry and stratigraphic conditions encountered in practice.

The variational calculus method uses the application of Euler's equations to directly compute the shape and location of the critical slip surface. Numerous research papers have been written and there has been controversy regarding the analysis. This paper will review some of the history of its application to slope stability problems. Hopefully, further research will prove the method to be of greater use in engineering practice.

**VARIATIONAL CALCULUS METHOD**

The variational calculus method uses the calculus of variations to compute a minima value for functional. The functions considered are force and/or moment equilibrium equations where the factor of safety is minimized while satisfying several other conditions. Although a number of solutions have been published to date, the method still requires further research. The method was primarily developed over the past 2 decades and some of the earlier researchers are as follows: (Kopacynski, 1961; Dorfman, 1965, 1970; Garber, 1973a, 1973b; Chen and Snithab, 1975; and Biernatowski, 1976). During the past decade, most of the research appears to have been conducted in Spain (Revilla, 1976; Revilla and Castillo, 1977; Luceno and Castillo, 1980), Israel (Baker and Garber, 1977a, 1977b, 1978) and India (Narayan, 1976; Narayan, Bhatkar and Ramamurthy, 1976, 1978). Necessary to note is that no exhaustive literature review on the variational calculus method. Rather, recent research is outlined and the variational calculus method is described. In particular, there is interest in the prediction of the location and shape of the critical slip surface.

The variational calculus method can be performed without any assumption regarding the shape of the slip surface or the normal stress distribution along the slip surface. The only assumption is that the slip surface is continuous. The failure criterion is also assumed to be Mohr-Coulomb. The method seeks a lower bound solution in the sense that no kinematic considerations are used in the solutions. However, it is possible to find references to the variational calculus method as an upper bound solutions, a lower bound solution or an "exact" solution. There are some differences in the variational calculus methods presented in the literature.

The formulation first consists of writing force and moment equilibrium equations similar to those for limit equilibrium methods. The equilibrium equations are written in a definite integral form and are later approximated as a summation when being solved. The equilibrium equations are solved for the factor of safety and term safety functionals (Baker and Garber, 1977b). The safety functions must be minimized using a variational analysis. In addition, a variety of boundary conditions must be satisfied as i) the natural or geometric boundary conditions, ii) the stress boundary conditions, iii) the transversality conditions and iv) the necessary and sufficient conditions for the absolute minimum (Narayan and Ramamurthy, 1980). The transversality condition must be satisfied in order to locate the starting and ending positions for the critical slip surface. Euler's equations must be satisfied in order for the functional to be a minimum. Euler's first differential equation allows the determination of the shape and location of the critical slip surface. Euler's second equation allows the determination of the normal stress along the critical slip surface.
The variational calculus method still requires a trial and error type solution for the minimum factor of safety. However, there is an increased efficiency in solving problems of the design nature.

In 1977, Castillo and Revilla proposed a variational calculus method in which the limit equilibrium formation was similar to Janbu's simplified method without the empirical correction, $f_o$, to account for the interslice shear forces. Results are presented for a cohesive soil. The computed factors of safety were significantly lower than values from the friction-circle method (Taylor, 1937), upper bound solution. Similar results were also published in Geotechnique (Revilla and Castillo, 1977). Two discussants suggested that the empirical correction factor, $f_o$, should be applied to the solution (Justo, 1978; Friedli and Giger, 1978). The correction factor brings the comparison into closer agreement. However, the significance of this observation must be questioned since total equilibrium has not been rigorously satisfied.

Baker and Garber (1977b) presented a variational calculus method for a soil with both cohesive and frictional components. Pore-water pressures were also taken into account. Horizontal, vertical and moment equilibrium are satisfied. Three integral equilibrium equations, two geometric condition equations and two variational boundary conditions are solved. The critical slip surface is shown to have the form of a log-spiral based on the solution of Euler's equations. In 1978, Baker and Garber concluded that the log-spiral form is the solution when the rotational mode of failure occurs. However, it is noted that it is also possible to have a straight line slip surface for a transition. Both means. It is assumed for both modes must be computed. The solutions are for a homogeneous, isotropic case. In general, the slip surface may consist of both linear and log-spiral shapes due to varying soil conditions. Although the pre-determined shape was not assumed for the slip surface, it must be noted that the special cases considered in the derivation cannot be applied to heterogeneous conditions. The special cases considered show that the computed factor of safety is not a function of the normal stress distribution on the slip surface. For this reason it is not necessary to make a further assumption regarding the interslice forces. A scheme for solving for the critical slip surface and minimum factor of safety is suggested but no results are presented.

Ramamurthy, Narayan and Bhatkar (1977) also presented a variational calculus method in terms of the effective strength soil parameters. The interslice forces are divided into effective and pore-water pressure components. The interslice shear component is assumed to be a constant function of the interslice effective normal force, similar to Spencer's assumption (Spencer 1967). An equation is also written for the vertical factor of safety between each slice. Both moment and force equilibrium equations are written in a definite integral form. The minimum factor of safety is computed in a direct manner using the Ritz technique. The critical slip surface is computed but the normal stress distribution on this slip surface are not computed. An example problem involving a homogeneous embankment is solved. The embankment is approximated by a polynomial of degree 4. In another paper (Narayan and Ramamurthy, 1978) the same problem is solved and further results are presented. The variational calculus method gave a factor of safety of 0.71 while Spencer's method gave a value of 0.77. The factors of safety are for the most critical slip surface in each case. Further similar comparisons of the critical factors of safety and the shape of the slip surface would be useful.

In 1980, Luceno and Castillo expressed reservations about the application of the variational calculus methods to slope stability analysis. Their concern was that "sufficient conditions" may not be satisfied. As a result, the computed factor of safety may be neither a relative or absolute minimum value. The method proposed by Chen and Shitban (1978) and Baker and Garber (1977b, 1978) is based on global equilibrium (i.e., the interslice forces are not considered) and the solution is said to be unbounded and can go to zero. The method proposed by Narayan, Bhatkar and Ramamurthy (1978, 1977) is based on local equilibrium (i.e., an assumption is made regarding the interslice forces) and does not satisfy enough conditions for the functional to be a minimum. Narayan and Ramamurthy (1980) disagreed that none of the variational calculus methods "satisfy the conditions for a functional to be a minimum". The state that "For a functional to be a minimum it is not enough if only Euler's condition is satisfied. It has to further satisfy necessary and sufficient conditions to prove the existence of such a minimal is a global minimal". They state that the Lagrange's necessary condition and the Weierstrass' sufficient condition for a global minimum must also be satisfied.

The potential for the use of the variational method in solving slope stability problems is obvious. However, the method requires further research. Further studies should be performed on the shape of the critical slip surface in cohesive and frictional soils, and comparisons of the results should be made to conventional limit equilibrium methods satisfying total equilibrium. In order for the method to be of practical use, it must be also be applied to heterogeneous soil conditions. It must be possible to input the actual, irregular ground surface and stratigraphic conditions into the analysis.

**LIMIT ANALYSIS**

A complete limit analysis of a soil requires that three conditions be satisfied. These are the stress equilibrium equations, the stress versus strain relations and the compatibility equations relating strain and displacement. In the absence of satisfying all of these conditions it is possible to
satisfy only certain conditions. This gives rise to lower and upper bound solutions. The "upper bound" solution results in a higher factor of safety than for the complete solution. The reverse is true for a "lower bound" solution.

The lower bound solution satisfies the equilibrium equations, the stress boundary conditions and the failure criterion for the soil. The resulting stresses yield a statically admissible stress field for the problem. The lower bound solution normally gives no consideration to kinematics. The soil approaches the failure condition as a result of applying a load (Sokolovski, 1960). It is possible, however, to factor down the strength of the soil and thereby reduce the applied load to zero for slope stability problems. Classically this method has been used primarily for bearing capacity type problems.

The upper bound solution satisfies velocity boundary conditions, strain and velocity compatibility conditions and the failure criterion of the soil. This results in a kinematically admissible velocity field. The velocity field can also be thought of as a displacement field if all processes are assumed to be independent of time. The stress conditions are defined only in the region that is deforming and they need not be in equilibrium. Interfaces within the deforming mass must also be at the yield conditions. The shear strength parameters can be reduced by a constant called the factor of safety. The upper bound solution can be obtained by the energy approach or the equilibrium approach, both giving the same results. The equilibrium approach is of the most interest in solving slope stability problems. It makes use of the equilibrium of forces associated with the assumed kinematically admissible failure mode. The condition of kinematic acceptability is best satisfied by the construction of a velocity (or displacement) hodograph.

Limit methods of slope stability analysis can be broadly classified as upper and lower bound solutions. Respectively, these methods are similar to statical and kinematical methods (Chen, 1975).

The limit equilibrium method common to slope stability analysis is an upper bound type of solution since it specifies a simplified mode of failure and applies statics. Figure 6 shows the manner in which the displacement field or mode of failure is commonly specified. Although limit equilibrium methods adopt the general philosophy of an upper bound solution, they do not meet all the precise requirements. Limit equilibrium methods fall short of a complete solution in that no consideration is given to kinematics and that equilibrium conditions are only satisfied in a limited sense. For example, failure conditions are not ensured at the interslice surfaces. From a limit analysis standpoint, limit equilibrium methods do not necessarily provide an upper or lower bound solution.

**FIG. 6 Kinematic Conditions Assumed in Limit Equilibrium Analyses (From Izbicki, 1981)**

Izbicki (1981) terms the results from a limit equilibrium methods as a "reduced" upper bound. Although the term "reduced" upper bound may not be completely acceptable from a plasticity viewpoint, the point is that the limit equilibrium factor of safety will be slightly less than the upper bound factor of safety. In his paper, a theory is outlined for an upper bound solution by the method of slices. The assumption is made that the interslice shear forces are at their limit values. In other words, the same factor of safety is applied between the slices as at the base of a slice. Horizontal, vertical and moment equilibrium equations are written for each slice. The condition of internal force equilibrium is then written for the entire sliding mass. The upper bound factor of safety is determined by solving the above equations. Further details on the actual procedure for solving the equations would prove useful. The same upper bound factor of safety is also confirmed by the construction of a velocity hodograph and making use of the equation of total work balance.

Using the above method, Kisiel, Izbicki, Skoczylas and Stilger-Szydzio (1981) present results to show the relationship between factors of safety. For the example considered, the computed upper bound factor of safety was 1.45 with the velocity hodograph being satisfied. Comparisons were made with the simplified Bishop and Janbu's Generalized methods which gave factors of safety of 1.42 and 1.37, respectively. In the limit equilibrium methods, the state of stress on the interslice surface lies below the limit state. The most extreme case is to completely ignore the interslice forces as in the case of the Fellenius or Ordinary method. The factor of safety then drops to 1.30. This is still an upper bound type of solution.

**FINITE ELEMENT METHOD (DISPLACEMENT TYPE METHOD)**

The finite element method is an extremely powerful computational tool that has developed rapidly over the past two decades. The finite element method theoretically satisfies the necessary requirements for a complete solution of a slope stability problem. However, there are restrictions that have limited its use in practice. For example, the complete...
nonlinear stress versus strain relationship for the soil is presently difficult if not impossible to accurately evaluate. Relating the nonlinear stress versus strain relationship to the strength parameters of the soil has resulted in a more useful form for practical applications.

The finite element method produces a distribution of stresses and displacements which must be interpreted and applied in design in a different manner than the results from a limit equilibrium type of analysis. For this reason, attempts have been made to select potential slip surfaces and compute a type of safety factor. Chen and Chameau (1982) have used this procedure for both two- and three-dimensional analysis. The mean factor of safety, \( F_{\text{mean}} \), is defined as a weighted average of shear strength over shear stress.

\[
F_{\text{mean}} = \frac{\Sigma (c + \sigma_n \tan \phi) \, dA}{\Sigma r_n \, dA}
\]

where:
- \( c \) = cohesion intercept
- \( \phi \) = internal angle of friction
- \( \sigma_n \) = normal stress
- \( r_n \) = shear stress
- \( dA \) = area at the bottom of a vertical column

An example problem showed that the limit equilibrium method and the finite element method gave similar results. Pore-water pressures were not considered. The finite element method yielded slightly higher factors of safety. The difference was only 1% for the two-dimensional case and 6% for the three-dimensional case. The limit equilibrium analysis used the assumption of a constant inclination of the interfacial forces (i.e., Spencer’s assumption) for both the two- and three-dimensional cases.

Early research using the two-dimensional finite element method (Wright, Kulhawy and Duncan, 1973) compared the normal force at the base of a slice to normal forces computed from a limit equilibrium analysis. More recently the finite element method has been used to estimate the stress distribution along the interfacial surface (Wilson and Fredlund, 1983; Fan, 1983). The shear and normal stresses are integrated along vertical surfaces. The interfacial shear force is then divided by the interfacial normal force to get a direction for the interfacial resultant. The results are used to obtain a more realistic side force function for limit equilibrium analyses satisfying total equilibrium.

THREE-DIMENSIONAL LIMIT EQUILIBRIUM ANALYSIS

The remainder of this paper is devoted primarily to aspects of limit equilibrium analysis. Generally the stability of a slope is analysed in a two-dimensional, plane strain manner. However, there are many situations where the configuration of the sliding mass is highly three-dimensional. The engineer must be concerned about the difference in factor of safety between the two- and three-dimensional cases.

Azzouz and Baligh (1975) showed that for a cohesive soil, the three-dimensional, end effects can lead to a 4 to 40 percent increase in the factor of safety. Certainly errors associated with the assumption of two-dimensionality can be at least as significant as variations between various two-dimensional limit equilibrium methods.

Prior to 1977, three-dimensional stability analyses were limited to cohesive soils. Hovland (1977) presented a three-dimensional analysis for soil with cohesion and frictional strength. Pore-water pressures were also not considered. The results were analysed; i) a cone-shaped shear surface on a vertical slope, and ii) a wedge-shaped shear surface. It was shown that three-dimensional factors of safety (i.e., \( F_3 \)) were always higher than comparable two-dimensional factors of safety (i.e., \( F_2 \)). This was true for cohesive soils. It was also shown that this need not always be the case for cohesionless soils. The ratio of \( F_2/F_3 \) was shown to be quite sensitive to magnitude of cohesion and angle of friction, as well as to the geometry of the three-dimensional slip surface.

Azzouz, Baligh and Ladd (1981) performed back analyses on 4 embankment failures using two- and three-dimensional, undrained shear strength analyses. Their results indicated that the end or edge effect (i.e., the addition of the third dimension) increased the two-dimensional factors of safety by a maximum of 30 percent and a minimum of 7 percent. Therefore, the results from two-dimensional, back analyses will overestimate the strength and can lead to an unsafe situation when these values are used in design.

Chen and Chameau (1982) presented a three-dimensional, limit equilibrium method that is similar in formulation to Spencer’s two-dimensional method. That is, the interfacial forces are assumed to be at a constant inclination throughout the soil mass. The soil can have cohesion and a friction angle, and a pore-water pressure can be taken into account. Figure 7 shows the isometric and frontal view of the spoon-shaped slip surface. The moment and force equilibrium equations are solved to give the inclination of the resultant interfacial forces and the factor of safety. Their results can be summarized as follows: i) the three-dimensional effects are more significant when the length of the sliding mass is small; ii) for gentle slopes, the two-dimensional effects are more significant for soil with a high cohesion intercept and a high angle of internal friction; iii) the three-dimensional factor of safety may be less than the two-dimensional factor of safety for soils with a low cohesion intercept and a high angle of internal friction.
friction; and iv) the pore-water pressure may cause the three-dimensional effects to be even more significant.

![Isometric View of 3-Dimensional Sliding Mass](image1)

**FIG. 7a** Isometric View of 3-Dimensional Sliding Mass

![Frontal View of Sliding Mass](image2)

**FIG. 7b** Frontal View of Sliding Mass

**FIG. 7** Isometric and Frontal Views of Spoon Shaped Sliding Mass (From Chen and Chameau, 1982)

**NONLINEAR SHEAR STRENGTH ENVELOPE**

The term, factor of safety, $F_s^e$, is generally used in a consistent manner for all types of limit equilibrium analysis. The shear strength of a saturated soil is defined by a linear Mohr-Coulomb failure criterion. In terms of effective stress, the equation is as follows:

$$
T = c' + (u_n - u_w) \tan \phi'
$$

(2)

where:

- $T$ = shear strength
- $c'$ = effective cohesion intercept
- $u_w$ = pore-water pressure
- $\phi'$ = effective angle of internal friction.

To ensure equilibrium conditions along a slip surface, the shear strength parameters along the entire slip surface are reduced by a constant (i.e., $c'/F_s^e$ and $\tan \phi'/F_s^e$). This is an oversimplification which has met with surprising success.

The actual shear strength envelope is often curved, particularly in the lower normal stress range (Figure 8). It is common practice to approximate the failure envelope by a straight line tangent to the average normal stress range applicable to the slip surface under consideration. Numerous empirical nonlinear failure envelope equations have been proposed (Maksimovic, 1979). Attempts have also been made to use a bilinear shear strength envelope.

There are difficulties in using a bilinear or nonlinear shear strength envelope in the assessment of the normal stress (or force) at a particular location on the slip surface. In limit equilibrium methods, the normal force is generally computed by summing forces in a vertical or near vertical direction. The shear strength parameters appear in the derived equation. If the shear strength envelope is nonlinear, an interactive approach must be adopted to compute the corresponding normal stress and shear strength parameters.

Maksimovic (1979) described the treatment of a nonlinear failure envelope with a zero cohesion intercept. The secant angle of friction is represented as an empirical, three-parameter expression. The limit equilibrium method used to solve for the factor of safety uses similar statics and assumptions as that of the Morgenstern-Price method (1965). The computational scheme is summarized as follows. For the first iteration, the normal stresses along the slip surface are computed assuming that all interslice forces are zero (i.e., Fellenius or Ordinary method assumption). A hierarchy is established for satisfying the nonlinear requirements of the analysis and a factor of safety is computed satisfying both moment and force equilibrium. After convergence is achieved with respect to the overall equilibrium equations, a new average normal stress along the base of each slice is computed and compared with previous normal stress values. The analysis in which the new normal stresses are calculated is not completely clear from the paper. Presumably it is based on the summation of forces vertically and the previously computed interslice forces. If the two sets of normal stresses are within 1%, the solution is assumed to have converged. Otherwise, new angles of internal friction are computed and the computational scheme is repeated. An example problem is solved which demonstrates the stability of a granular slope depends upon its height.

Charles (1982) performed some circular
arc stability analysis on compacted rock fills and other soils with a non-linear failure envelope (Charles and Soares, 1984). The shear strength envelope was nonlinear and the strength varied as the normal stress raised to a power.

\[ \tau = A (\sigma_n - u_w)^b \]

where:
- \( A \) = intercept on a log-log plot of shear strength versus effective stress
- \( b \) = slope of a log-log plot of shear strength versus effective stress.

The equation has limits to the normal stress range over which it applies. The slope stability equation derived satisfies the same conditions as the simplified Bishop method. The results presented for a homogeneous rock fill show that the factor of safety is a function of height and the depth of the critical slip surface is dependent on the nonlinearity of the failure criterion. Charles and Soares (1984) present stability charts for a range of simple slopes.

A planar, three-dimensional form of the Mohr-Coulomb criterion has been proposed for unsaturated soils (Fredlund, Morgenstern and Widger, 1978; Fredlund, 1979).

\[ \tau = c' + (\sigma_n - u_a) \tan \phi' + (u_a - u_w) \tan \phi_b \]  

(3)

where:
- \( u_a \) = pore-air pressure
- \( \phi_b \) = friction angle with respect to the \((u_a - u_w)\) stress variable

The equation can be graphically presented in a Mohr-Coulomb form with the cohesion as a function of matric suction (Figure 9) (Ho and Fredlund, 1982). Conventional methods of slope stability analysis can be used for analysing unsaturated soil slopes if equation [3] is rearranged as follows.

\[ \tau = c + (\sigma_n - u_a) \tan \phi' \]  

(4)

where
- \( c = c' + (u_a - u_w) \tan \phi_b \)

The cohesion of the soil, \( c \), is now a function of the negative pore-water pressures. This equation has been applied to the analysis of steep, residual soil slopes in Hong Kong (Ching, Sweeney and Fredlund, 1984). The analyses show that matric suction can significantly increase the computed factor of safety and also affect the location of the critical slip surface.

EARTHQUAKE ANALYSIS

There are two approaches commonly used to analyse slopes subjected to earthquake conditions; namely, i) the pseudo-static analysis, and ii) the dynamic analysis. In the pseudo-static analysis, the effect of the earthquake is accounted for by a constant static equivalent of the seismic force. The force is generally applied in the horizontal direction only and acts through the centroid of the slice. Although there are limitations to this approach it still remains the common approach particularly for small dams. A dynamic analysis is performed using the finite element technique. The analysis gives an indication of displacements and stresses at any point, with respect to time. The difficulty with the dynamic analysis lies in obtaining accurate input data.

Sarma (1979) used a new limit equilibrium method for calculating the factor of safety of a soil mass subjected to earthquake loadings. A horizontal force is applied at the centroid of each slice. A plot of factor of safety versus acceleration factor can be presented from the analysis (Figure 10). This form of presentation is also applicable to other limit equilibrium methods of analysis. An assumption is made regarding the relative magnitude of the interslice shear force. An attempt is made to bring all interslice surface within the soil mass into a limit state. The slices are not vertical but it is suggested that wedge-shaped slices are more reasonable. In this respect, the analysis is most applicable to actual failed slopes (Chowdhury, 1980).

![FIG. 9 Extended Mohr-Coulomb Envelope for Unsaturated Soils (From Ho and Fredlund, 1982)](image)

![FIG. 10 Relationship between Factor of Safety F and Acceleration Factor K (From Sarma, 1979)](image)
It must be realized that there is difficulty in assessing the magnitude of internal shear strengths (Chowdhury, 1981). The analysis remains a limit equilibrium method and should not be confused with a critical equilibrium procedure. It is suggested that the acceleration factor should be used as a measure of safety.

Martins (1979) expressed some reservations, stating that Sarma’s method gives unrealistic acceleration factors in some cases and suggested a modification to the method. Martins, Reis and Matos (1981) suggest that the conventional factor of safety be used as a measure of safety for cohesionless soils whereas the acceleration factor is satisfactory for cohesive slopes.

Chugh (1982) proposed a limit equilibrium method to account for earthquake forces. The method is similar to Spencer’s method with respect to the statics satisfied and the assumptions regarding the interslice force directions. The slip surface can be nonlinear in shape. The earthquake force can be horizontal or inclined and is applied through the center of mass of a slice. The factor of safety is solved for various acceleration factors. The acceleration factor corresponding to a factor of safety of one is termed the coefficient of yield acceleration. The equations of motion (i.e., Newmark method) are used to estimate displacements along a segmented potential slide mass. The mass is assumed to behave as a rigid plastic system. The time history for the accelerations produced by an earthquake is then selected for design. Whenever the earthquake acceleration exceeds the yield acceleration, relative displacement of the slice begins. The displacement calculations make use of the linear acceleration method and the time history of an earthquake acceleration in double integration of the differential equation of motion. While the displacements are approximate, they are an upperbound solution.

PROGRESSIVE FAILURE

Considerable research has been presented on progressive failure and retrogressive landslides during the past 7 years. The stress versus strain curve for the soil is typically of a strain softening form dropping significantly from a peak value to a residual shear strength. Attention is given herein to suggested changes in limit equilibrium methods for accommodating progressive failure.

Normally a constant factor of safety is assumed to apply for the entire slip surface during a limit equilibrium analysis. Law and Lumb (1978) suggested a limit equilibrium method of slices that considers two sets of shear strength parameters for the soil (i.e., peak strength parameters and post-peak or residual strength parameters). The analysis satisfied both moment and force equilibrium and makes an assumption regarding the direction of the interslice forces. A criterion is established for the initiation of local failure in the slope. The factor of safety is computed using peak strength parameters and a check is made for local failure. During the check for local failure, it is assumed that the resultant of all interslice forces acting on a slice is zero. Once the local failure takes place, the strength parameters are abruptly reduced to their post-peak (i.e., residual) values. The decrease in strength on one slice leads to a transferring of force to neighboring slices or the process of progressive failure. The final factor of safety had to be re-defined as the ratio of the overall available strength after the propagation of local failure to the actual shear strength required for equilibrium. Three well-documented case records were examined by Law and Lumb (1978). Although there was difficulty in analysing these records using conventional limit equilibrium methods, they were successfully analysed by the proposed method employing post-peak strengths. The analyses showed that initial local failure was found to occur both at the top and near the toe of the slope.

Haug, Sauer and Fredlund (1977) used the lateral pressure theory in conjunction with a limit equilibrium approach to back-analyse various blocks of a retrogressive landslide. The lateral earth pressure was found to range between the active and at-rest pressure conditions. No design procedure was outlined.

Other methods proposed for the analysis of progressive failure problems can be classified as extensions to the limit equilibrium approach. Generally use is made of the strain softening, stress versus strain curve. Athanasium (1980) used the summation of forces horizontal and vertical to satisfy equilibrium and assumed horizontal interslice forces. Progressive failure was assumed to be initiated from the top of the slope and proceed to the bottom. Interactions between horizontal forces are computed for each side of the slice at the top of the slope. The initial shear stress at the base of each slice was calculated from the force equilibrium equation. The mobilized shear stress along the base of the sliding surface was then written as a function of the shear modulus of the soil. As a result, the displacement of each slice was calculated, integrating from the top of the slope.

Bernander and Olafsson (1981) also suggested the measurement of the complete stress versus strain curve for the analysis of progressive failure in strain softening soils. They suggest that progressive failure is strongly affected by factors such as brittleness, deviatoric strain at peak strength and the initial stress level. The proposed analysis utilizes the stress versus strain curve and helps explain the bulges or passive zones often noticed on the ground surface. It is suggested that parameters other than factor of safety be characterized for a slope.

Foerster and Georgi (1981) attempt to extend the analysis of stability in strain softening soils to include time effects. A finite element model is used to study the progression of failure as a function of time. The evaluation of reliable soil parameters is
necessary and may be difficult to evaluate.

Botsiropoulos and Cavounidis (1981) reported irregularities in the evaluation of peak and post-peak strength parameters. They showed that in many cases the shear stress passed to the strain softening side of the stress versus strain curve without ever reaching the expected peak shear strength. This also has implications on the modelling of progressive failure.

The published research shows that no common consensus or modelling procedure has as yet emerged for analysing progressive failure problems. Extensions of models beyond a limit equilibrium analysis require soil parameters that are difficult to assess. Proposed extensions require measurements of the complete stress versus strain curve and the lateral earth pressures in the soil. There certainly is a need for further well-documented case histories.

GENERAL LIMIT EQUILIBRIUM THEORY

The various limit equilibrium analyses remain the main methods used in the assessment of the stability of a slope. Even with ready access to digital computers, the limit equilibrium approach has remained in common usage due to its simplicity and flexibility in handling practical problems. There have also been reservations raised against the limit equilibrium approach (Tavenas, Trak and Leroueil, 1980; Chowdhury, 1981). Factors such as the history of slope formation, initial state of stress and details concerning the landslide movement are not addressed in the analysis. However, success in the usage of limit equilibrium methods of analysis has been commendable. At the same time, the limit equilibrium approach must not be considered infallible. Basically, limit equilibrium is not fundamental to phenomena concerning stability (Chowdhury, 1980).

Most limit equilibrium methods divide the soil mass above an assumed slip surface into a number of vertical slices. The vertical slices are assumed to be of infinitesimal width in the Morgenstern-Price and Janbu's Generalized methods (Figure 11). The derived factor of safety equations take the form of infinite integrals. It is presently impossible to write equations for the geometry, soil and pore-water pressure conditions of interest in practical problems. Therefore, in order for these equations to be solved, the sliding mass must be divided into slices of discrete width. Most other methods of slices (e.g., Ordinary or Fellenius, Simplified Bishop, Spencer, GLE and others) derive the factor of safety equations commencing with vertical slices of a discrete width (Figure 12). The resulting equations take on a summation form. The difference in formulation is merely of mathematical preference and need not influence the computed factor of safety. Sarma (1979) in his formulation used slices or wedges with the sides inclined from the vertical. This shape of slice may have particular significance for the back-analysis of failed earth masses.

![FIG. 11 Forces Acting on a Slice of Infinitesimal Width](image1)

![FIG. 12 Forces Acting on a Slice of Finite Width](image2)

Theoretical and quantitative comparisons have been previously made between the various limit equilibrium methods. This paper will merely attempt to summarize these findings. The general equilibrium method (i.e., GLE) (Fredlund and Krahm, 1977; Fredlund and Pufahl, 1981) is used to show the theoretical relationship between the various methods.

A slip surface of arbitrary shape is shown in Figure 12. Once the shape of the slip surface is assumed, a kinematic constraint is imposed on the movement of the soil mass. Although these kinematic constraints are not directly used in the limit equilibrium analysis, the resulting factor of safety will be of the upper bound type.

The forces on each slice are defined as follows:

- \( W \) = total weight of a slice of width 'b' and height 'h'
- \( P \) = total normal force acting on the base of a slice, equal to \( g \cdot F \)
- \( F \) = the normal stress on the base of the slice and 'l' is the length of the slip surface at the base of a slice.
- \( S_m \) = mobilized shearing resistance at the base of a slice
- \( E_L, E_R \) = total interslice normal force on the left and right sides of a slice, respectively
- \( X_L, X_R \) = interslice shear force on the left and right sides of a slice,
\[ a_L, a_R = \text{resultant external hydrostatic force at the left and right ends of the assumed slip surface, respectively} \]

\[ R = \text{radius or moment arm associated with the mobilized shearing resistance,} S' \]

\[ x = \text{horizontal distance from the centroid of each slice to the centre of moments} \]

\[ f = \text{offset distance from the normal force to the centre of moments, and} \]

\[ a_L, a_R = \text{perpendicular distance from the resultant external hydrostatic force to the arbitrary centre of moments.} \]

The definition of factor of safety is the same for all methods; namely, the factor by which the shear strength must be reduced to bring the sliding mass into equilibrium. The mobilized shear force on the base of a slice, \( S_m \), can be written in terms of the effective stress, Mohr-Coulomb failure criterion.

\[ S_m = \frac{T_i}{F} = \frac{1}{F} [c' + (P - u)] \tan \phi' \]

where:

\[ f = \text{shear strength} \]

\[ c' = \text{effective cohesion parameter} \]

\[ \phi' = \text{effective angle of internal friction} \]

\[ u = \text{pore-water pressure, and} \]

\[ F = \text{factor of safety.} \]

Additional conditions generally imposed are that i) the factor of safety for the cohesive component of strength are equal for all soils, and ii) the factor of safety is the same for all slices.

The elements of statics that can be used to derive the factor of safety are the summation of forces in two directions and the summation of moments. These elements of statics, along with the failure criteria, are not sufficient to make the slope stability analysis determinate as is shown in Table I.

There are 4n equations and (6n - 2) unknowns and therefore (2n - 2) independent assumptions must be made to render the analysis statically determinate. If more than (2n - 2) assumptions are made, more unknowns must be introduced in order to solve for the factor of safety. Alternatively, some of the original assumptions could be changed from one iteration to another. In reality there are many assumptions that can be made but mention will be made only of those most commonly used. Most limit equilibrium methods start by making an assumption regarding the point of application of the normal force at the base of a slice. Usually the normal is assumed to act through the center of the base. This reduces the excess of unknowns over knowns to (n - 2).

Different limit equilibrium methods also choose not to satisfy all elements of statics. The elements of statics satisfied by some of the common methods of slices is shown in Table II. The assumptions used to render each of the analysis determinate are summarized in Table III. There are many possible sets of interslice forces that will satisfy equilibrium conditions. Therefore, it is useful to have further admissibility criteria that are also satisfied (Chowdhury, 1980). For example, i) there should not be tension within the freebody, ii) the criterion of failure should not be violated along any of the interslice boundaries, and iii) the effective normal stresses along the slip surface should be compressive. Even when these conditions are satisfied, the interslice force system need not be unique.

Let us consider the manner in which some of the common limit equilibrium methods are made statically determinate (Sarma, 1979). In Bishop's simplified method, n assumptions are made for the point of application of the normal force, \( P \), and (n - 1) assumptions are made for the magnitude of the interslice shear forces (i.e., \( X = 0 \)). This results in one more assumption being made than required. As a result, the horizontal force equilibrium on one slice
TABLE II
Elements of Statical Equilibrium Satisfied by Various Limit Equilibrium Methods

<table>
<thead>
<tr>
<th>METHOD</th>
<th>FORCE EQUILIBRIUM</th>
<th>MOMENT EQUILIBRIUM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st Direction (e.g., Vertical)</td>
<td>2nd Direction (e.g., Horizontal)</td>
</tr>
<tr>
<td>Ordinary or Fellenius</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Bishop's Simplified</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Janbu's Simplified</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Spencer</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Chugh</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Morgenstern-Price</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>GLE</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Corps of Engineers</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Lowe and Karafiath</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Sarma</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

* Any of two orthogonal directions can be selected for the summation of forces.
** Moment equilibrium is used to calculate interslice shear forces.

cannot be satisfied with the computed factor of safety. Originally it was shown that the error was small when the slip surface was circular (Bishop, 1955). Later, it was shown that the error was also small for noncircular (or composite type) slip surface provided a reasonable center for moment equilibrium was selected (Fredlund and Krahn, 1977).

TABLE III
Assumptions in Limit Equilibrium Methods

<table>
<thead>
<tr>
<th>METHOD</th>
<th>ASSUMPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordinary or Fellenius</td>
<td>Interslice forces are neglected</td>
</tr>
<tr>
<td>Bishop's Simplified</td>
<td>Resultant interslice forces are horizontal (i.e., there are no interslice shear forces).</td>
</tr>
<tr>
<td>Janbu's Simplified</td>
<td>Resultant interslice forces are horizontal; an empirical correction factor, ( f_i ), is used to account for interslice shear forces.</td>
</tr>
<tr>
<td>Janbu's Generalized</td>
<td>Location of the interslice normal force is defined by an assumed line of thrust.</td>
</tr>
<tr>
<td>Spencer</td>
<td>Resultant interslice forces are of constant slope throughout the sliding mass.</td>
</tr>
<tr>
<td>Morgenstern-Price</td>
<td>Direction of the resultant interslice forces is defined using an arbitrary function. The percentage of the function, ( \lambda ), required to satisfy moment and force equilibrium is computed.</td>
</tr>
<tr>
<td></td>
<td>Direction of the resultant interslice forces is defined using an arbitrary function. The percentage of the function, ( \lambda ), required to satisfy moment and force equilibrium is computed.</td>
</tr>
<tr>
<td>Corps of Engineers</td>
<td>Direction of the resultant interslice force is ( l ) parallel to the ground surface, or ( l ) equal to the average slope from the beginning to the end of the slip surface.</td>
</tr>
<tr>
<td>Lowe and Karafiath</td>
<td>Direction of the resultant interslice force is equal to the average of the ground surface and slope of the base of each slice.</td>
</tr>
<tr>
<td>Sarma</td>
<td>The shear strength is assumed to be mobilized on the sides of all slices. The inclination of slice interfaces is varied to produce a critical condition.</td>
</tr>
</tbody>
</table>

* Chugh's method (1982) is the same as Spencer's method but a constant acceleration force acts on each slice.
In Janbu's Generalized method, n assumptions are made concerning the point of application of the normal force, P, and (n-1) assumptions are made concerning the point of application of the interstage normal forces. Sarna (1979) points out that this is one more assumption than required and the method should not technically be called a rigorous solution. He points out that the position of the last normal, P, is not used and therefore moment equilibrium on the last slice is not satisfied. The moment equilibrium equation is used to compute the interstage shear forces and can be visualized as generating an interstage force function.

The Morgenstern-Price method (1965) makes n assumptions for the point of application of the normal force, P, and (n-1) assumptions regarding the relative relationship between the magnitudes of the interstage forces. Since one more assumption is made than required, an extra unknown, $\lambda$, is introduced. This method rigorously satisfies static equilibrium.

Spencer's method (1967) and the GLE method also rigorously satisfy static equilibrium. Spencer's method was originally derived for circular slip surfaces but has since been extended to noncircular slip surfaces (Spencer, 1973). The GLE method also applies to noncircular slip surfaces and is used to further compare limit equilibrium methods.

The normal force, P, acting on the base of each slice is derived from the summation of forces in a vertical direction. This is the same for most limit equilibrium methods (Fredlund and Krahn, 1977).

\[
P = \frac{W - (X_R - X_L) - c'F \sin \alpha + u'F \tan \theta' \sin \alpha}{m_{\alpha}}
\]

where:

\[
m_{\alpha} = \frac{\cos \alpha + \sin \alpha \tan \theta'}{F}
\]

Two independent factor of safety equations can be derived, one with respect to moment equilibrium and the other with respect to horizontal force equilibrium. Moment equilibrium can be satisfied with respect to an arbitrary point above the central portion of the slip surface. For a circular slip surface, the center of rotation is an obvious center of moments. The center of moments is immaterial when both force and moment equilibrium are satisfied. When only moment equilibrium is satisfied, the computed factor of safety varies slightly with the point selected for the summation of moments (Boulton, 1977).

\[
F = \frac{\sum (c' R + (P - u) R \tan \theta')}{\sum Wx - \sum Pf + Aa}
\]

where:

\[
F_m = \text{factor of safety with respect to moment equilibrium.}
\]

The factor of safety with respect to force equilibrium is derived by summing the forces in the horizontal direction for all slices.

\[
F_f = \frac{\sum (c' \cos \alpha + (P - u) \tan \theta' \cos \alpha)}{\sum Pf \sin \alpha + Aa}
\]

where:

\[
F_f = \text{factor of safety with respect to force equilibrium.}
\]

The summation of horizontal forces on each slice can be used to compute the total interstage normal force, E.

\[
(X_R - X_L) = [W - (X_R - X_L) \tan \alpha - \frac{S_m}{\cos \alpha}]
\]

The assumption is made that the interstage shear force, X, is related to the interstage normal force, E, by a mathematical function (Morgenstern and Price, 1965).

\[
X = \lambda f(x)E
\]

where:

\[
f(x) = \text{a functional relationship which describes the manner in which the magnitude of X/E varies across the slip surface, and}
\]

\[
\lambda = \text{a scaling constant which represents the percentage of the function, } f(x), \text{ used for solving the factor of safety equations.}
\]

The GLE formulation makes it possible to readily visualize the relationship between the various limit equilibrium methods (Fredlund and Krahn, 1977; Fredlund, Krahn and Pufahl, 1981). For example, the moment equilibrium factor of safety equations specializes to the Bishop's Simplified method when the interstage shear forces are set to zero (i.e., $\lambda = 0.0$). Similarly, the force equilibrium factor of safety equation specializes to Janbu's Simplified method when the interstage shear forces are set to zero. The computed factor of safety must be multiplied by the correction factor, f, Factors of safety computed by the Spencer's and Morgenstern-Price methods can also be duplicated using GLE, as the condition where both force and moment equilibrium have been satisfied for a specified function. The Corps of Engineers and Lowe-Karafiath methods can be solved using the force equilibrium equation and the appropriate side force function.
Figure 13 shows a plot of factor of safety versus $\lambda$ for a simple 11 metre high, 2:1 slope (Fredlund, Krahn and Pufahl, 1981). The soil properties are as follows: total unit weight = 18.8 kN/m$^3$; angle of internal friction = 20 degrees; cohesion = 28.75 kPa and pore-pressure ratio = 0.2. The interslice force function was assumed to be a constant. Otherwise interslice force functions can significantly affect the slope of the force equilibrium factor of safety versus $\lambda$ plot. The moment equilibrium factor of safety is quite insensitive to the interslice force function. As a result, any method satisfying moment equilibrium will have relatively consistent factors of safety. This does not, however, apply to the Ordinary method since it ignores all interslice forces.

![Figure 13](image1)

**FIG. 13** Comparison of Factors of Safety for Example No. 1 (Circular Slip Surface)
(From Fredlund, Krahn and Pufahl, 1981)

Figure 14 shows a plot of factor of safety versus $\lambda$ for a similar slope with a composite slip surface (i.e., part circular and part linear). The lower portion of the slip surface has a cohesion reduced to zero and an angle of internal friction of 10 degrees. The relationship between the factors of safety remains the same for a circular slip surface. The plot also shows that the factor of safety from Bishop's Simplified method will remain close to that satisfying complete equilibrium, even for a composite slip surface.

**FIG. 14** Comparison of Factors of Safety for Example No. 2 (Composite Slip Surface)
(From Fredlund, Krahn and Pufahl, 1981)

iteration and is often used to obtain an estimate of the factor of safety for solving other methods. The factor of safety obtained by the Ordinary method is lower than values from other methods. Convergence of other nonlinear factor of safety equations is more readily ensured when the factor of safety is decreased monotonically to its final value (Soriano, 1976). For this reason, the Ordinary method factor of safety is often arbitrarily increased prior to being used for other methods.

When only one factor of safety equation must be solved (e.g., Bishop's Simplified, Janbu's Simplified, Corps of Engineers, Lowe-Karafiath), it is not necessary to have an elaborate technique to solve for the factor of safety. For example, the Newton-Raphson technique can be used to solve the Bishop's Simplified equation; however, a simple iterative technique produces rapid convergence.

When more than one factor of safety equation must be solved (e.g., Spencer, Morgenstern-Price, GLE), it is necessary to use a more elaborate technique to solve for the factor of safety. The simplest form of solution is that presented by Spencer (1967) where each factor of safety equation is solved independently and plotted versus an indicator of the side force function. This is similar to Figures 13 and 14 and the solution can be obtained graphically. Fredlund (1974) used a least squares, best-fit, second order polynomial for each factor of safety versus $\lambda$ equation and then solved for the intersection point. The procedure was termed the "best-fit regression" technique.

In 1981, Fredlund presented a "rapid solver" technique which required much less computations to find the factor of safety.
satisfying both force and moment equilibrium (Figure 15). The initial \( \lambda \) value was estimated and approximately three iterations were conducted for the force and moment factor of safety equations. The \( \lambda \) value was decreased or increased depending on the relative magnitude of the moment and force equilibrium factors of safety. The intersection point was then estimated and calculations proceeded until the difference between the moment and force equilibrium factors of safety were within the required tolerance.

\[
\delta = \frac{3F_f}{m} \frac{3F_f - F_m}{3F_f} \frac{3F_f}{m} \frac{3F_f - F_m}{3F_f} \frac{3F_f}{3 \lambda 3 F_f - 3 F_m 3 \lambda}
\]

Modified or extended forms of the Newton-Raphson technique have also been used to solve for the factor of safety (Chen and Morgenstern, 1982).

NUMERICAL DIFFICULTIES IN SOLVING FOR THE FACTOR OF SAFETY

Two difficulties have been expressed concerning the solution of nonlinear factor of safety equations; namely, i) multiplying of solutions, and ii) non-convergence of the solution. Ching (1981) reported the existence of more than one numerical solution when solving for the factor of safety satisfying both moment and force equilibrium. The multiplicity of solution was not due to the insufficiency of assumptions but the nonlinear nature of the equations. However, Sarma (1981) stated that "Of all these values of factor of safety, only the largest one gives shear forces at the base of the slice in the direction which is implicit in the limit equilibrium method. All other solutions are therefore not acceptable".

Some convergence difficulties have been reported in solving all the nonlinear factor of safety equations. Most concern has been expressed over the solution of Janbu's Generalized method and suggestions have been made to assist in obtaining a converging solution (Janbu, 1980).

Soriano (1976) analyzed the nonlinear factor of safety equations from a mathematical standpoint and outlined limiting conditions that should be considered when devising iterative schemes. If these conditions are not adhered to, divergence or convergence to parasitic roots may occur. Stable and convergent iterative schemes are reported for some of the limit equilibrium methods. It was also reported that force equilibrium methods are more sensitive to the side force inclination than are the moment equilibrium methods.

Ching and Fredlund (1983) showed that most problems associated with non-converging solutions can be traced to one of three possible conditions. First, the shape of the assumed slip surface may be unrealistic. Second, high cohesion values may result in a negative normal force and produce instability. Third, the assumption used to render the analysis determinate may impose unrealistic conditions and prevent convergence.

Related to the first condition, it is noted that all nonlinear factor of safety equations contain the term, \( m_q \). Difficulties with convergence most commonly occur as a result of the selection of a slip surface that is inconsistent with the earth pressure theory (Figure 16). As a result, the \( m_q \) term goes to zero or approaches zero.
in a negative or positive sense. This causes the normal force at the base of a slice to tend to infinity. The net result is that the normal on one or more slices may have an unreasonably large effect on the factor of safety. The suggested remedy is to ensure that the shape is such that active and passive theoretical considerations with respect to direction are not violated at the scarp and toe of the slope, respectively.

Nonconvergence can be encountered in any limit equilibrium method which uses an unrealistic assumption regarding the interslice condition. This problem appears most often in Janbu's Generalized method (Ching, 1981). The moment equilibrium equation is used to generate the equivalent of an interslice force function based on an assumption for the line of thrust. The shape of the resulting function can be unrealistic when compared to an elastic analysis (Fan, 1983) of the mass (Figure 17). The steepness of the function at the ends produces high interslice shear which may exceed the weight of the slice. It is suggested that the assumption used in the analysis should be somewhat consistent with the stresses in the soil mass.

SIDE FORCE FUNCTIONS

With the exception of Janbu's Generalized method, most limit equilibrium methods make an assumption regarding the direction of the resultant interslice forces (see Table 3). The Bishop Simplified method assumed the slope of the resultant interslice force was zero; as did Janbu's Simplified method. Spencer's method allowed the selection of arbitrary constant slopes for the direction of the resultant interslice forces. In the Morgenstern-Price method, any one of a number of arbitrary functional relationships could be used to define the direction of the resultant interslice force. The Corps of Engineers method related the direction of the resultant interslice forces to the average surface slope. The Lowe-Karafiat method also took into account the shape of the slip surface.

Morgenstern and Price (1967) suggested that the interslice force function should be related to the shear and normal stresses on vertical slices through the soil mass. In 1979, Maksimovic made use of the finite element method and a nonlinear characterization of the soil to compute the stresses in a soil mass. These stresses were used to compute an interslice force function for computing the factor of safety.

Wilson and Fredlund (1983) and Fan (1983) performed a detailed study on interslice force functions computed from finite element analysis. Both linear and nonlinear stress versus strain relations were assumed (Duncan and Chang, 1970). The differences between the linear and nonlinear analyses were shown to be small when no elements were stressed beyond the failure conditions. Figure 18 shows a typical result for a 3 horizontal to 1 vertical slope and Figure 19 shows the results for a deep seated slip surface on a 1 horizontal to 2 vertical slope. Based on a large number of analyses, a general, empirical interslice force function was proposed.

$$f(x) = Ke^{(-CN \omega)}$$

(13)

where:

- $e$ = base of the natural logarithm
- $K$ = magnitude of the interslice force function at mid-slope (i.e., maximum value)
- $C$ = variable to define the inflection points
- $N$ = variable to specify the flatness or sharpness of curvature
- $\omega$ = dimensionless x-position relative to the midpoint of the slope.
The variable, $K$, is a function of the slope inclination and the depth factor (Figure 20). The constants 'C' and 'N' are related to the slope inclination (Figure 21). These parameters were computed for a circular slip surface. Analyses for composite or noncircular slip surfaces showed variations in magnitude for some of the constant in equation [13] but the general shape of the function remained the same. Figures 22 and 23 show plots of equation [13] while varying C and n. The inflection points on the function agree closely with the scarp and toe coordinates.

**FIG. 18** The Inter-slice Side Force Ratio (X/E) Distribution for a Deep Seated Slip Surface in the Three Horizontal to One Vertical Slope (From Wilson and Fredlund, 1983)

**FIG. 19** The Inter-slice Side Force Ratio (X/E) Distribution for a Deep Seated Slip Surface in the One Horizontal to Two Vertical Slope (From Wilson and Fredlund, 1983)

**FIG. 20** The Inter-slice Side Force Ratio, $K$, at Midslope Versus the Depth Factor, $D$

**FIG. 21a** Values of "C" Versus Slope Angle

**FIG. 21b** Values of "N" Versus Slope Angle

**FIG. 21** C and N Constants for Various Slope Angles
FIG. 22 Interslice Side Force Function When Varying the C Constant

FIG. 23 Interslice Side Force Function When Varying "n" Constant

Equation [13] controls only the relative magnitude of the resultant interslice forces. When the interslice function is used in the GLK analysis, the value approached 1 for the cohesionless soil case (Figure 24). The computed lines of thrust fall near the 1/2 point and no problems have been observed with convergence (Figure 25).

FIG. 24 Variation of the Lambda Values with Cohesion and Angle of Internal Friction (From Fan, 1983)

FIG. 25a The Interslice Side Force Ratio Distribution for a Deep-Seated Slip Surface in a 3:1 Slope

FIG. 25b The Resulting Line of Thrust for the Deep-Seated Slip Surface in a 3:1 Slope

FIG. 25 Typical Interslice Force Function and Computed Line of Thrust (From Fan, 1983)

QUANTITATIVE COMPARISON OF FACTORS OF SAFETY

Numerous studies have been done to compare the factors of safety computed by the different methods. Janbu (1980) showed a comparison of factors of safety for simple slopes composed of soils with both cohesive and frictional components of strength. His conclusion was definite and stated as follows: "The differences are so small that the procedures compared are numerically equal for all practical purpose". More details on the quantitative relationships between the limit equilibrium methods published by Duncan and Wright (1980) and can be summarized as follows:

i) the Ordinary or Fellenius method always gives conservative factors of safety because the calculated normal force is low. In cases of a flat slope with high pore-water pressures, the error in the computed factor of safety may be as high as 50%. When there are no pore-water pressures, the error is not more than 10%.

ii) Methods that satisfy all conditions of equilibrium give essentially the same value for the factor of safety. Deviations between the various methods should not be more than ±5%.

iii) Although the Bishop Simplified method does not satisfy horizontal force equilibrium, it gives virtually the same factor of safety as methods that satisfy complete equilibrium. The checks were made
for circular slip surfaces.

iv) Methods that satisfy only force equilibrium conditions can be expected to give factors of safety within ±15% of those satisfying complete equilibrium. When the inter slice forces are assumed to be parallel to the ground surface, the factors of safety can deviate even further.

Ching and Fredlund (1984) showed the "m" and "n" stability coefficients for several of the commonly used limit equilibrium methods. The results confirm that the computed factors of safety are essentially the same for those methods which satisfy complete equilibrium. Deviations in stability coefficients (i.e., m and n) are 1 to 5%. For toe circles, the differences in the m and n values obtained by the various methods are minimal. Maximum differences are approximately 10% when comparing the Ordinary and Simplified Bishop method. For deeper slip surfaces, the differences in the m and n coefficients between the Ordinary and Bishop Simplified method range from 5 to 20% (Figures 26 and 27). Janbu's Simplified method differs from the Bishop Simplified method by 5 to 10%; Janbu's Simplified factors of safety being higher. The differences in the "m" and "n" coefficients increase with an increase in the angle of internal friction and slope inclination.

Under certain conditions the results from different methods yield the same answers. For the case of a cohesive soil, all methods satisfying moment equilibrium (or moment and force equilibrium) give the same results. For a Division of a semi-infinite slope, all limit equilibrium methods give the same factor of safety when the soil is cohesionless. The factor of safety for methods satisfying force equilibrium depend on the inter slice force function.

Presently the problem of obtaining consistent answers for the factor of safety from different computer programs is more serious than the difference between methods resulting from differences in theory. Lumadaine and Tang (1982) studied the results from 54 solutions to 29 sample problems. The results were not analysed statistically but the initial submissions of results show ranges of factor of safety in the order of ±8% from the mean. Obviously there is need for further research in the area of factor of safety calculations.

REFERENCES


