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Benefits of adaptive automatic mesh refinement

Igor Petrovic, Murray Fredlund

Semi-automatic mesh generation is a time-consuming and error-prone process. This is particularly true for engineering computations where the mesh requires varying levels of complexity. This paper studies two numerical models that produce converged solutions with the assistance of automatic adaptive mesh refinement, AMR. The studies illustrate how automated adaptive mesh refinement can reduce modeling time as well as errors during the modeling process. The AMR solutions were performed using the SVFlux / FlexPDE software. The results are discussed in the contexts of the solutions published by Chapuis (2012). Chapuis (2012) analyzed the same example problems while using user-controlled mesh design when performing the numerical solutions.

Types of errors that occur in finite element analysis

The mathematical type of errors introduced into the finite element solution of a given differential equation can be attributed to three basic sources (Reddy, 2006):

1. domain approximation errors –due to approximation of domain,
2. quadrature and finite arithmetic errors – these are errors due to the numerical evaluation of integrals and the numerical computation on a computer,
3. approximation errors – these are errors due to the approximation of the solution through interpolation functions.

This list does not consider errors in programming, and differences between the numerical model and the real physics. For more complete list see for example Oberkampf et al. (1995) and Roache (2009).

Convergence

The main problem in any numerical model which needs to be addressed consist of the questions of *how good the approximation is and how it can be systematically improved to approach the exact answer*. The answer to the first question presumes knowledge of the exact solution.

The second question can be answered from studies in interpolation theory. The finite element approximation is known to converge in the energy norm when $\|e\| < Ch^p$, for $p > 0$, where h is the distance between nodes on a uniform mesh (the characteristic element length), p is called the rate of convergence. The rate depends on the degree of the polynomial used to approximate true solution u the order of the highest derivative of u in the weak form, and whether there are local singularities in the domain. The constant, C, is independent of u and will be influenced by the shape of the domain and whether Dirichlet or Neumann boundary conditions are employed. Typically $p = k + 1 - m > 0$ where k is the degree of the highest complete polynomial used in the interpolation and m is the order of the highest derivative of u in the weak form. The above equation for the error would be a straight line plot for a log-log plot of error versus mesh size. In that case the slope of the line is the rate of convergence, p (Akin, 2005).

Finite element adaptive mesh refinement, AMR

An adaptive mesh refinement procedure measures the adequacy of the mesh and refines the mesh wherever the estimated error is large. The system iterates the mesh refinement and solution until a user-defined error tolerance is achieved. The most common criterion in general engineering use is that of prescribing a total limit of the estimated error computed in the energy norm as described in previous chapter. Often this estimated error is specified to not exceed a specified percentage of the total norm of the solution. An adaptive mesh refinement procedure is used to reduce estimated errors once a finite element solution has been obtained. The procedure is referred to "adaptive" since the process depends on previous results at all stages.

Various procedures exist for the refinement of finite element solutions. Broadly these fall into two categories (Zienkiewicz et al., 2005).

1. The *h*-refinement in which the same class of element continues to be used but it is changed in size, in some locations while being made larger in some locations and smaller in others, to provide maximum economy in reaching the desired solution,
2. The *p*-refinement in which the same element size is used and there is a simple increase, generally hierarchically, in the order of polynomial used in the definition of the elements.

It is occasionally useful to divide the above categories into subclasses, as the *h*-refinement can be applied and thought of in different ways. Three typical methods of *h*-refinement are:

1. Element subdivision – if existing element show too large an estimated error, the elements are simply divided into smaller elements while keeping the original element geometry boundaries intact,
2. Mesh regeneration (remeshing) – on the basis of a given solution, a new element size is predicted in all the domains and a totally new mesh is generated,
3. *r*-refinement – keeps the total number of nodes constant and adjusts their position to obtain an optimal approximation. This method is difficult to use in practice and there is little reason to recommend its usage.

The *p*-refinement subclasses are:

1. one in which the polynomial order is increased uniformly throughout the entire domain,
2. one in which the polynomial order is increased locally while using hierarchical refinement.

Occasionally it is efficient to combine the *h*- and *p*- refinements and call it the *hp*- refinement. In this procedure both the element size and the polynomial degree, *p* is altered.

Advantages of using automatic adaptive mesh generators (numerical examples)

Advantages of using automatic adaptive mesh generators are illustrated through comparison of results obtained on the numerical models analyzed by Chapuis (2012). Chapuis (2012) presented two examples problems where he created finite element meshes semi-automatically and solved the seepage problems. The same example problems were solved using automatic mesh refinement using the SVFlux / FlexPDE finite element code.

Cut-off example

The geometry of the model (i.e., dam with partial cut-off wall; $k_{sat,\ homogenous\ soil} = 8.13\times10^{-3}$ m/day) analyzed is presented in Figure 1.

In the reference article, convergence of the solution was obtained using a uniform mesh with an element size of 0.5 m. From Figure 1 it can be seen that the converged solution obtained when using the automatic adaptive mesh refinement has larger elements in most parts of the analyzed domain. The exception is found around the cut-off wall where the element size is significantly smaller than the overall average element size. For the mesh presented in Figure 1, the calculated flow rate was 6.82×10^{-7} m³/s. Calculation time for the mesh presented at Figure 1 was 0.01 minutes. Comparison of results obtained with manually-controlled meshes and automatic-controlled adaptive meshes are presented in Figure 2. Calculation computational times associated with using a disabled mesh generator with a specified maximum element size of 0.5 m, increased to 7.37 minutes while the flow rate solution remained the same (note that an older computer was used for this study). The consequence of further reductions in the element size to 0.3 m was an increased calculation time from 7.37 minutes to 36.03

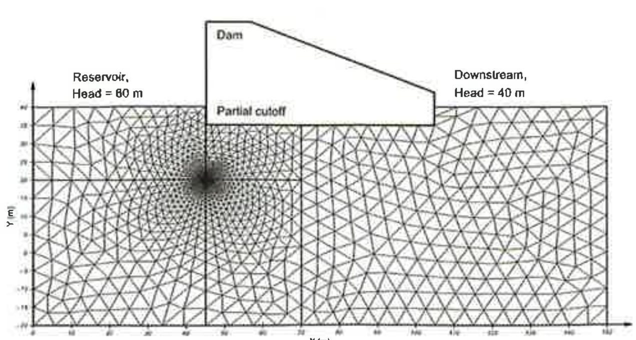

Figure 1. Partial cut-off wall model geometry with mesh generated using the automatic adaptive mesh generator.

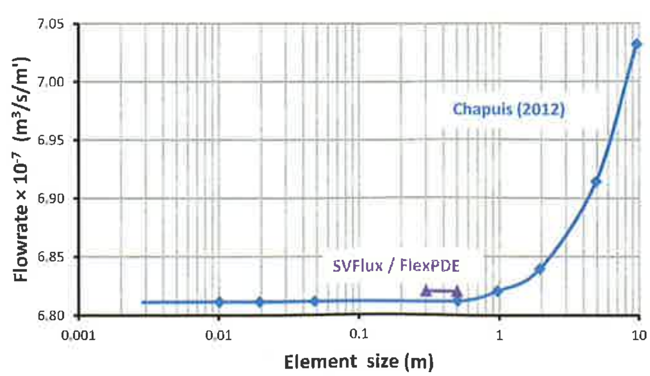

Figure 2. Converged leakage flow-rate for the cut-off example.

Figure 3. Pumping well (confined aquifer) model geometry with mesh generated by adaptive mesh generator; take a note that few triangles have an angle higher than 90 degrees, which means a poor shape for calculations (axisymmetric problem, radius of confined aquifer was 600 m).

minutes while the flow rate remained unchanged.

Confined aquifer example

The geometry of the second model (i.e., pumping well in confined aquifer; ksat, homogenous soil = 4.0×10-4 m/s) is presented in Figure 3.

In the reference article (Chapuis, 2012) the solution converged using a uniform mesh with an element size of 0.1 m. From Figure 3 it can be seen that converged solution obtained with use of the automatic adaptive mesh has larger elements in most parts of the analyzed domain, except around the pumping well where the element size is significantly smaller than the overall average (0.2 m in average). For the mesh presented in Figure 3 the computed flow-rate was 369.17 m^3/day. The calculation time for the mesh presented in Figure 3 was 0.02 min.

Comparison of flow-rate and total head obtained when the mesh was manually-controlled and when the mesh was automatically generated using an adaptive mesh generator is presented in Figure 4 and Figure 5. Calculation time for disabled mesh generator with a specified maximum element size of 0.1 m (in the region which covers area from pumping well to the 20.15 m in the radial direction), has increased to 1.15 minutes while the flow-rate solution decreased to 366.54 m^3/day. The size of the elements in the remainder of the domain was 1 m.

Conclusion

In the reference article author stated that finer grid provides a more accurate solution. However, the solutions converged only after the mesh was refined to an element size of 0.5 m (in the cut-off example) and 0.1 m (in the confined aquifer example), uniformly distributed across the problem domain. From Figure 1 and Figure 3 it can be seen that mesh obtained when using automatic adaptive mesh generators can have much larger elements in most parts of the domain while the accuracy of the solution is preserved as presented in Figure 2, Figure 4 and Figure 5.

Chapuis (2012) suggested the creation of a final confirmation/verification mesh (i.e., a finer mesh) to verify that solution has actually converged (this is done to define the true solution as accurately as possible when closed-form solution is unknown). It was also stated that this final verification step might be a time consuming process (for long transient problems computing time can take hours or even days). For lengthy, transient problems, it was suggested that final verification mesh could be omitted in order to save time. With use of automatic adaptive mesh refinement generators, this final verification step is not necessary since the mesh generator refines the mesh in various parts of the domain until the solution converges within user specified tolerance limits. Since the accuracy of the solution depends on these tolerance limits, it is necessary that user have a clear understanding of the finite element method when using adaptive mesh generators in an efficient manner. It can't be emphasized enough that it is the engineer who must check the numerical tools and their solutions. However, is should be also noted that for the default error limits should result in a converged solution for most standard geotechnical problems defined in Eurocode 7 as Geotechnical Category 1 and 2.

In summary, automatic an adaptive mesh generator can also result in the following benefits.

- A optimized (locally finer and locally coarser) mesh means fewer number of equations,

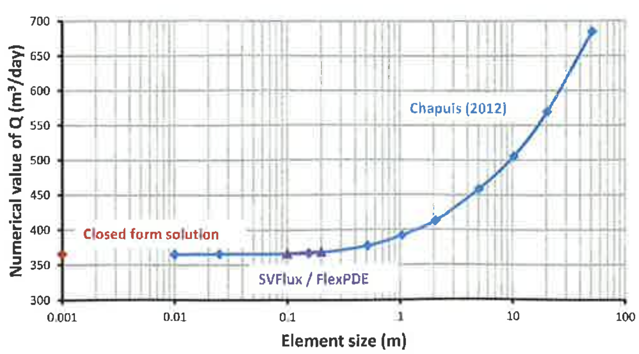

Figure 4. Converged numerical flow-rates.

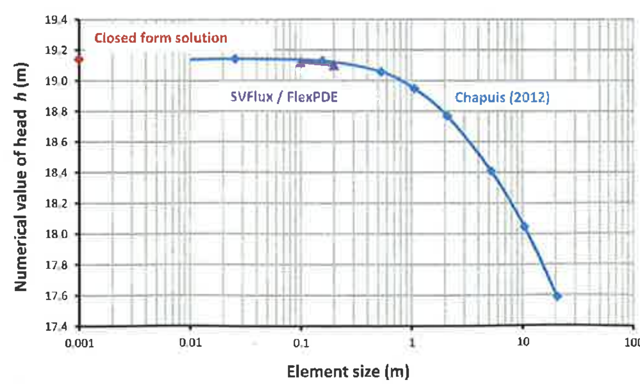

Figure 5. Converged total heads at r = 20.15 m.

- Lesser number of equations means less time needed for calculation,
- Less time needed for calculation means that employee productivity is increased.

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